DEHN SURGERY EQUIVALENCE RELATIONS ON THREE-MANIFOLDS

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§**1. Introduction.** Suppose M is an oriented 3-manifold. A *Dehn surgery* on M (defined below) is a process by which M is altered by deleting a tubular neighborhood of an embedded circle and replacing it again via some diffeomorphism of the boundary torus. It was shown by W.B.R. Lickorish [Li] and A. Wallace [Wa] that any closed oriented connected 3-manifold can be obtained from any other such manifold by a finite sequence of Dehn surgeries. Thus under this equivalence relation all closed oriented 3-manifolds are equivalent. We shall investigate this same question for more restricted classes of surgeries. In particular we shall insist that our Dehn surgeries preserve the integral (or rational) homology groups. Specifically, if M_0 and M_1 have isomorphic integral (respectively rational) homology groups, is there a sequence of Dehn surgeries, each of which preserves integral (respectively rational) homology, that transforms M_0 to M_1? What is the situation if we further restrict the Dehn surgeries to preserve more of the fundamental group? Is there a difference if we require "integral" surgeries? We also show that these Dehn surgery relations are strongly connected to the following questions concerning another point of view towards understanding 3-manifolds. Is there a Heegard splitting of M_0, $M_0 = H_1 \cup_f H_2$ (H_i are handlebodies of genus g and f is a homeomorphism of their common boundary

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surface), and a homeomorphism g of ∂H_1 such that M_1 has a Heegard splitting using $g \circ f$ as the identifiation? Since there are many natural subgroups of the mapping class group, such as the Torelli subgroup and the "Johnson subgroup," one can ask the same question where g is restricted to lie in one of these subgroups. This is related to work of Morita on Casson's invariant for homology 3-spheres [Mo]. Even under these restrictions it has been known for some time that any homology 3-sphere is related to S^3. This fact has been used to define, calculate and understand invariants of homology 3-spheres (such as Casson's invariant) by choosing such a "path to S^3" in the "space" of 3-manifolds.

We shall show that, in general, the equivalence relations among 3-manifolds thus induced are non-trivial but most can be beautifully characterized in terms of classical invariants such as the linking form, the cohomology ring and Massey products. This precise and beautiful correspondence between the geometric equivalence relation of Dehn surgery and the algebraic equivalence of classical invariants will form the philosophical basis of a new theory of finite type invariants for 3-manifolds by T.D. Cochran and P. Melvin [CM].

Before stating the main theorems let us be more specific. We suppose throughout that M is a compact oriented 3-manifold. Suppose that γ is a smoothly embedded oriented circle in M which is of finite order in $H_1(M;\mathbf{Z})$. Let $N(\gamma)$ denote a regular neighborhood of γ. We define several (isotopy classes of) embedded closed curves on $\partial N(\gamma)$. The *meridian of* γ, $\mu(\gamma)$, is given by the boundary of a meridional disk $* \times D^2 \subset S^1 \times D^2 \equiv N(\gamma)$. It is easy to see that $\mu(\gamma)$ is unique and is of infinite order in $H_1(M - N(\gamma);\mathbf{Z})$. A *parallel* of γ, denoted $\rho(\gamma)$, is any simple closed curve which is homologous to γ in $N(\gamma)$ and intersects $\mu(\gamma)$ precisely once. Any two parallels "differ" by some number of meridians. The longitude of γ, $\ell(\gamma)$, is a simple closed curve which is homotopic in $N(\gamma)$ to some *positive* integral multiple of γ and which is of finite order in $H_1(M - N(\gamma);\mathbf{Z})$. The longitude is unique and, in the case that γ is null-homologous, constitutes a *preferred parallel*. However in general $\ell(\gamma) \cdot \mu(\gamma)$ is not 1 and hence the longitude (unfortunately)

may not be used as a parallel. With respect to *some* choice of parallel (the preferred parallel if γ is null-homologous) one defines p/q-Dehn surgery on M along γ, denoted M_γ, to be $[M - \dot{N}(\gamma)] \cup_\phi (S^1 \times D^2)$ where $\phi : S^1 \times \partial D^2 \longrightarrow \partial N(\gamma)$ is an orientation-reversing homeomorphism which sends the curve $* \times \partial D^2$ to a curve representing $p\mu(\gamma) + q\rho(\gamma)$ in $H_1(\partial N)$. The number p/q is called the *framing* of γ. We assume $(p, q) = 1$ and do not allow $q = 0$. We call the surgery *integral* if $q = \pm 1$ and note that this is independent of the choice of parallel. The surgery is *longitudinal surgery* if $p\mu + q\rho$ coincides with the longitude up to sign. If γ is null-homologous this is equivalent to $p = 0$. The resulting Dehn surgery M_γ does not depend on the orientation of γ. Then we can ask whether or not two 3-manifolds are related by Dehn surgeries on curves γ which are not arbitrary, but are restricted to lie in some subgroup N of $\pi_1(M)$. In addition we shall restrict the "framing" p/q of the surgery in order that the integral (respectively rational) homology groups are preserved. Specifically, suppose N is the normal closure of a finite number of elements of $G = \pi_1(M)$ and also suppose that the elements of N are trivial in $H_1(G; \mathbf{Z})$ (or $H_1(G; \mathbf{Q})$ in the $\mathbf{Q}$-case). We shall be concerned primarily with these examples:

i) $N = G_k$, the k-th term of the lower central series of G, where $G_{k+1} = [G, G_{k-1}]$ and $G_1 = G$;

ii) $N = \{e\}$, the trivial group;

iii) $N = G''$, the second derived group of G, that is $G'' = [G_2, G_2]$;

iv) $N = G_2^{\mathbf{Q}} = \{x \in G \mid \exists n, x^n \in G_2\}$ which is the set of elements which are torsion in $H_1(M; \mathbf{Z})$ (this would be in the rational case).

v) $N = G_k^{\mathbf{Q}}$ the k^{th} term of the rational lower central series where $G_{k+1}^{\mathbf{Q}}$ is generated by $[G, G_k^{\mathbf{Q}}]$ and elements of G for which some power lies in $[G, G_k^{\mathbf{Q}}]$ [Stallings].

In all these cases N is a characteristic verbal subgroup and we can view N as a *functor* from groups to normal subgroups. That is, given a space X we can speak of $N(\pi_1(X))$ in each of these cases.

Definition 1.1: M_1 is N-surgery related to M_0 if there is a finite sequence $M_0 = X_0$, $X_1, \ldots, X_m = M_1$ where X_{i+1} is obtained from X_i by p_i/q_i Dehn surgery along γ_i with $\gamma_i \in N(\pi_1(X_i))$ and $p_i = \pm 1$ (in the $\mathbf{Z}$ case where γ_i is null-homologous and p_i is well-defined) and merely non-longitudinal surgery (in the $\mathbf{Q}$-case).

In particular, if N is the "k^{th} lower central series subgroup" as in i), then we will say that M_1 is *k-surgery equivalent to* M_0. In the next section we will see that, in fact, the k-surgery relation is an equivalence relation, and indeed preserves $\pi_1(M)/N_0$.

The case $k = 2$, perhaps most fundamental, we call "integral homology surgery equivalence" or sometimes merely "surgery equivalence". The question of characterizing this equivalence relation was posed by G. Kuperberg via a newsgroup posting and was partially answered by the Ph.D. thesis of the second author Gerges. A more precise answer is given in this paper. We find (see Theorem 3.1) that this equivalence relation is completely controlled by not just by $H_1(M)$ but by the full triple cup product structures and by the linking form on the torsion subgroup of $H_1(M)$. For example, we deduce that any 2 closed, connected, oriented 3-manifolds with isomorphic $H_1 \cong \mathbf{Z}^m$ $(m < 3)$ are (integral homology) surgery equivalent. This generalizes the well-known result for homology spheres $(m = 0)$. The latter certainly appeared in public lectures by Andrew Casson in 1985, and we are informed that it was known earlier. This appeared in a 1987 paper by S.V. Matveev, where it was also announced that two 3-manifolds have isomorphic H_1 and linking forms if and only if they are related by "Borromean surgeries" [Ma, Theorem 2 and Remark 2].

Other sample results concerning integral homology surgery equivalence are:

Corollary 3.5. *Let $\mathcal{S}_m$ be the set of surgery equivalence classes of closed oriented 3-manifolds with $H_1 \cong \mathbf{Z}^m$. Then there is a bijection $\psi_* : \mathcal{S}_m \to \Lambda^3(\mathbf{Z}^m)/\operatorname{GL}_m(\mathbf{Z})$ from $\mathcal{S}_m$ to the set of orbits of the third exterior power of $\mathbf{Z}^m$ under the action induced by $\operatorname{GL}_m(\mathbf{Z})$ on $\mathbf{Z}^m$. Any such manifold is surgery equivalent to one which is the result of 0-framed*

surgery on a m component link in S^3 which is obtained from the trivial link by replacing a number of trivial 3-string braids by a 3-string braid whose closure is the Borromean rings.

Corollary 3.6. *A 3-manifold with $H_1 \cong \mathbf{Z}^m$ is surgery equivalent to $\#_{i=1}^m S^1 \times S^2$ if and only if its integral triple cup product form $H^1 \oplus H^1 \oplus H^1 \to \mathbf{Z}$ vanishes identically.*

Corollary 3.8. *If $H_1(M)$ is torsion-free then M is (integral homology) surgery equivalent to $-M$.*

Corollary 3.8 is interesting because it is not obvious from a geometric viewpoint how to construct such a path of surgeries, and the result fails in general for manifolds with torsion in H_1.

Corollary 3.9. *The set $\mathcal{S}(\mathbf{Z}_n)$ of (integral homology) surgery equivalence classes of closed oriented 3-manifolds with $H_1 \cong \mathbf{Z}_n$ is in bijection with the set of equivalence classes of units of $\mathbf{Z}_n$, modulo squares of units, the correspondence being given by the image of the fundamental class of the manifold in $H_3(\mathbf{Z}_n) \equiv \mathbf{Z}_n$. The correspondence is also given by the self-linking linking number of a generator of H_1 ($\lambda(1,1) = q$ and q is viewed as an element of $\mathbf{Z}_n$). The equivalence class of $q \in \mathbf{Z}_n$ contains the lens space $L(n,q)$.*

Corollary 3.10. *($H_1 \cong \mathbf{Z}_p$, p prime): Any 3-manifold with $H_1 \cong \mathbf{Z}_2$ is surgery equivalent to $\mathbf{R}P(3)$. For any odd prime there are precisely two surgery equivalence class represented by $L(p,1)$ and $L(p,q)$ where $q \not\equiv k^2 \mod p$. Hence if $p \equiv 3 \mod 4$ then we may take $q = -1$ and we see that $L(p,1)$ and $-L(p,1)$ are not surgery equivalent. If $p \equiv 1 \mod 4$ then $-L(p,1)$ is surgery equivalent to $L(p,1)$.*

Corollary 3.12. *If $\pi_1(M_0) \cong \pi_1(M_1)$ is abelian then M_0 is (integral homology) surgery equivalent to M_1 if and only if M_0 is orientation-preserving homotopy equivalent to M_1.*

Proposition 3.17. *Let M_0, M_1 be oriented, connected 3-manifolds. Then the following are equivalent:*

A. *M_0 is 2-surgery equivalent to M_1*

B. *There is a framed boundary link L in M_0 such that Dehn surgery (with framings ± 1) on L yields M_1.*

C. *M_0 and M_1 have isomorphic linking forms and triple cup product forms (in the sense of 3.1).*

Once again the result for homology spheres, where C is vacuous, was known. The earliest reference we can find is S.V. Matveev [Ma; Theorem A]. This result was reproved and used by S. Garoufalidis [Ga].

Finally let $\mathcal{K}$ be the subgroup of the mapping class group generated by Dehn twists along simple closed curves which bound a subsurface. Then, building on the observations of [GL] we have the following generalization of a theorem of Morita for homology spheres [Mo1; Proposition 2.3] [Jo1] [Mo2].

Theorem 3.18. *Let M_0, M_1 be closed oriented 3-manifolds. Then the following are equivalent.*

A. *M_0 and M_1 are 2-surgery equivalent.*

B. *There exist Heegard splittings $M_0 = H_1 \cup_f H_2$, $M_1 = H_1 \cup_{\psi \circ f} H_2$ where $\psi \in \mathcal{K}$.*

C. *M_0 and M_1 have isomorphic linking forms and triple cup product forms (as in 3.1).*

The rational case for $k = 2$ is controlled by $H_1(M; \mathbf{Z})/\mathrm{Torsion}$ and the *integral* triple cup product form (Theorem 5.1). Here are a few sample results concerning rational homology surgery equivalence.

Corollary 5.2. *If $m < 3$ any 2 closed, oriented 3-manifolds with identical first Betti number m are rational homology surgery equivalent.*

Corollary 5.3. *There is a bijection $\mathcal{S}_m^{\mathbf{Q}} \to \Lambda^3(\mathbf{Z}^m)/\operatorname{GL}_m(\mathbf{Z})$ given by the integral triple cup product form. Hence if M_0, M_1 have torsion-free homology groups then they are*

rational homology surgery equivalent if and only if they are integral homology surgery equivalent.

The situation for higher k is controlled by the above and also higher order Massey products (see §6).

Cases ii and iii above do not lead to equivalence relations and will be considered in a future paper.

This paper is organized as follows:

§1. Introduction

§2. Preliminaries

§3. Integral Homology Surgery Equivalence of Closed 3-Manifolds

§4. Proofs of Theorem 3.1 and other Basic Theorems

§5. Rational Homology Surgery Equivalence of Closed 3-Manifolds

§6. Surgery Equivalence Preserving Lower Central Series Quotients

§**2. Preliminaries.** In this section we prove several important technical results. In particular we show that the surgery relations having to do with the lower central series are in fact equivalence relations, but that symmetry fails in general. We also show that in these cases we may safely assume our surgeries are "integral" surgeries.

Proposition 2.1. *Suppose γ is a simple closed curve in M_0 which is null-homologous (or merely zero in $H_1(M_0;\mathbf{Q})$ in the $\mathbf{Q}$ case). Suppose M_1 is the result of $\pm 1/q$ surgery along γ (non-longitudinal surgery in the $\mathbf{Q}$ case) and γ' is the meridian of γ viewed as a curve in $\pi_1(M_1)$. If $\gamma \in \big(\pi_1(M_0)\big)_k$ for some $2 \le k < \omega$ then $\gamma' \in \big(\pi_1(M_1)\big)_k$ (in the $\mathbf{Q}$ case the same holds using the rational lower central series).*

Proof of 2.1. Let $N = N(\gamma)$ be the solid torus regular neighborhood of γ and T its boundary. First we treat the integral case. Let $G = \pi_1(M_0)$, $P = \pi_1(M_1)$, and suppose $\gamma \in G_k$. Then the longitude $\ell(\gamma)$ also lies in G_k and we know that there exists an immersed

$(k-1)$-stage half grope $f : (S_1, S_2, \ldots, S_{k-1}) \looparrowright M_0$ whose boundary is $\ell(\gamma)[FQ]$. Here we mean (as usual) that S_1 is a connected surface with $f\,\big|_{\partial S_1} = \gamma$ and that S_i is a collection of connected surfaces S_{ij} such that $f\,\big|_{\partial S_{ij}}$ is $f(a_{ij})$ where a_{ij} is a simple closed curve on S_{i-1}. Moreover each stage S_{i-1} has a $1/2$-rank system of such curves a_{ij} which occur as boundaries of S_i. We assume that the immersion of S_1, when restricted to a collar of S_1, is an embedding whose image lies in $M_0 - \dot{N}(\gamma)$. It follows that S_1 intersects γ transversely an algebraically zero number of times since $\ell(\gamma) = k\mu(\gamma)$ in $H_1\big(M_0 - \dot{N}(\gamma)\big)$ has only the solution $k = 0$. Recall that $\gamma \in G_2$ implies $\ell(\gamma)$ is null-homologous in $M_0 - \dot{N}(\gamma)$ and hence $\ell(\gamma) \in P_2$.

By general position we also assume that all a_{ij} lie in $M_0 - N$ and that each S_i meets γ transversely. Since $\gamma' = \mu(\gamma)$ is freely homotopic to $\pm q\ell(\gamma)$ in M_1, it follows that γ' lies in P_2 as well. Suppose now, by induction, that $\ell(\gamma) \in P_{n-1}$ for some $3 \le n \le k$. We shall show that $\ell(\gamma) \in P_n$ which will complete the proof in the integral case. Note that the induction hypotheses implies that $\mu(\gamma) \in P_{n-1}$. It follows that ∂S_{2j}, for any stage 2 surface, lies in P_{n-1}, because ∂S_{2j} bounds a $(k-2)$-stage half grope in M_0 which lies completely in $M_0 - \operatorname{int} N \subseteq M_1$ except for a collection of small 2-disks corresponding to the transverse intersections with γ. Hence ∂S_{2j} lies in P_{k-1} modulo a product of conjugates of $\mu(\gamma)$, which itself lies in P_{n-1}.

Now delete the algebraically-zero number of 2-disks of intersection of S_1 with N and tube along T to get S_1^* in $M_1 - \operatorname{int} N$. Then we see that $\ell(\gamma)$ is congruent, modulo P_n, to a product of conjugates of elements of the form $[x_i, \mu(\gamma)]^{\pm 1}$. Since $\mu(\gamma) \in P_{n-1}$, $\ell(\gamma) \in P_n$.

Now we address the "rational case." We suppose that $\gamma \in G_k^{\mathbf{Q}}$ and hence $\ell(\gamma) \in G_k^{\mathbf{Q}}$. Then there exists a "rational" $(k-1)$-stage half-grope whose "boundary" is $\ell(\gamma)$. By this we mean that $\partial S_1 = n_1\ell(\gamma)$ for some positive integer n_1 and similarly $\partial S_{ij} = n_{ij}a_{ij}$. Again we conclude that $S_1 \cdot \gamma$ is algebraically zero since the equation $n_1\ell(\gamma) = m\mu(\gamma)$ in $H_1(M_0 - \dot{N}(\gamma); \mathbf{Z})$ has only the solution $m = 0$ since $\ell(\gamma)$ is torsion while $\mu(\gamma)$ is not. We

claim that $\gamma' = \mu(\gamma)$ is "rationally related" to $\ell(\gamma)$ in P, that is that there exist integers x, y such that $\big(\mu(\gamma)\big)^x = \big(\ell(\gamma)\big)^y$ where $x \neq 0$. To see this, consider the (abelian) subgroup T of P generated by μ and the parallel ρ. Suppose $\ell(\gamma) = a\mu + b\rho$ where $(a, b) = 1$. Then T is a quotient of the abelian group $A = \langle \mu, \rho \mid p\mu + q\rho = 0\rangle$ and contains $\ell(\gamma)$. The vectors (p, q) and (a, b) are linearly independent in $\mathbf{Z} \times \mathbf{Z}$ since they are primitive and $(p, q) \neq \pm(a, b)$ since our surgery is not longitudinal. Hence $A/\langle \ell(\gamma)\rangle$ is a finite group and thus there are integers x, y $x \neq 0$ such that $x\mu = y\ell(\gamma)$ in A and hence in T. Using this relation, the proof now proceeds as in the integral case. $\square$

Corollary 2.2. *The relation of k-surgery equivalence is an equivalence relation on the set of oriented* 3*-manifolds. The relation of rational k-surgery equivalence is also an equivalence relation (Here we mean the rational lower central series and non-longitudinal surgery as in* example v)*.*

Proof of 2.2. Reflexivity and transitivity are obvious and symmetry is guaranteed by 2.1.

Proposition 2.3. *If M_0 and M_1 are k-surgery equivalent (respectively rationally k-surgery equivalent) then they are so equivalent using only* **integral** *surgeries, that is ± 1 surgeries (respectively integral non-longitudinal surgeries). In generality, if M_1 is N-surgery related to M_0 then there is an epimorphism $\pi_1(M_1) \twoheadrightarrow \pi_1(M_0)/N\big(\pi_1(M_0)\big)$ (in either $\mathbf{Z}$ or $\mathbf{Q}$ case of* Definition 1.1*). Consequently if M_1 is rationally* 2*-surgery equivalent to M_0 then $\beta_1(M_0) = \beta_1(M_1)$. If M_1 is integrally* 2*-surgery equivalent to M_0 then $H_1(M_0; \mathbf{Z}) \cong H_1(M_1; \mathbf{Z})$.*

Proof of 2.3. The sequence of homeomorphisms shown in Figure 2.4 using the "Rolfsen-Kirby" calculus is well-known (see [CG; p. 501] [R; p.]). This shows that $1/n$ surgery on γ is the same as a sequence of ± 1 surgeries on parallel copies of γ, denoted $\gamma_1, \gamma_2, \ldots, \gamma_n$. Let M_i^* be the result of surgery on $\{\gamma_1, \gamma_2, \ldots, \gamma_i\}$. We may view γ_{i+1} as the longitude of

γ_i and 2.1 guarantees that if γ_i lies in the k-th term of the lower central series of $\pi_1(M^*_{i-1})$ then $\mu(\gamma_i)$ and $\ell(\gamma_i)$ lie in $\big((\pi_1(M^*_i)\big)_k$. Hence M^*_n is k-surgery equivalent to M_0 via $+1$ surgeries as claimed.

FIGURE 2.4

Now consider the case that M_1 is rationally k-surgery equivalent to M_0 via a single surgery on γ in M_0. Since $M_0 = S^3_J$ for some framed link J in S^3, which may be assumed to be disjoint from γ, we may consider $\gamma \subset S^3$ with framing p/q with respect to the longitude of γ in S^3. Suppose the surgery is *not* integral, i.e., $q \neq 1$ (we assume $q > 0$). Then $p/q = \pm\big(m + \frac{r}{q}\big) = \pm\big((m+1) - \frac{q-r}{q}\big)$ where $m = [|p/q|]$ and $0 < r < q$. Then it is well known that the 3 pictures of Figure 2.5 are homeomorphic [Rolfsen,], where the upper sign is used if $p \geq 0$. Here the framings are all relative to S^3. Since $0 < r < q$ and $0 < q - r < q$, we may use 2.5 as the basis of an induction to reduce all surgeries in a daisy-chain of circles to integers (get $q = 1$). Since the first circle γ lies in $\big(\pi_1(M_0)\big)^{\mathbf{Q}}_k$, it suffices to show that the second curve in 2.5, say γ_2, lies in $G^{\mathbf{Q}}_k$ where $G = \pi_1(M^*)$, M^* being the result of $\pm m$ or $\pm(m+1)$ surgery on γ. In addition we must show that the $\pm m$ (or $\pm(m+1)$) surgery is integral and non-longitudinal and show the surgery on γ_2 is also non-longitudinal. Firstly, *at most one* of $\pm m$ or $\pm(m+1)$ could be longitudinal with respect to M_0 so we may choose the non-longitudinal one. Moreover the surgery on γ is integral means that the defining torus homeomorphism extends over a solid torus. But this is independent of coordinate system so the fact that $\pm m$, $\pm(m+1)$ are integers suffices to show these surgeries are integral relative to M^*.

FIGURE 2.5

Now, as has been mentioned, and is an immediate consequence of the second part of 2.3 (which is proved below), rational k-surgery equivalence preserves $H_1/(\text{torsion})$. Thus $\beta_1(M_0) = \beta_1(M^*) = \beta_1(M_1)$. This implies that the framing on γ_2 relative to M^* is non-longitudinal since it is precisely the longitudinal surgery which changes β_1.

In general, if M_1 is N-surgery related to M_0 via a single $\pm 1/n$ surgery then Figure 2.4 shows that there is a link L of n components in M_0, each component of which lies in $N\big(\pi_1(M_0)\big)$, such that adding n two-handles to $M_0 \times [0,1]$ along L yields a cobordism W between M_0 and M_1 rel $\partial M_0 = \partial M_1$. But then both inclusion maps induce epimorphisms on π_1 and the kernel of $\pi_1(M_0) \to \pi_1(W)$ lies in N. Thus $\pi_1(W)/N\big(\pi_1(W)\big) \cong \pi_1(M_0)/N\big(\pi_1(M_0)\big)$ and the second claimed result follows easily in the $\mathbf{Z}$ case. In the $\mathbf{Q}$-case, if M_1 is N-related to M_0 via a single non-longitudinal surgery then 2.5 shows that there is a cobordism W as above and a link L each of whose components lies in $N\big(\pi_1(M_0)\big)$ (in fact most are null-homotopic!). Then the argument above for the $\mathbf{Z}$-case holds. Note that if M_1 is rationally 2-surgery equivalent to M_0 then $\pi_1(M_1)$ maps onto the free abelian group $\mathbf{Z}^{\beta_1(M_0)} = \pi_1(M_0)/(\pi_1(M_0))_2^{\mathbf{Q}}$. Hence $\beta_1(M_1) \geq \beta_1(M_0)$. But by symmetry (2.1) $\beta_1(M_0) = \beta_1(M_1)$ and consequently $\pi_1(M_1)/(\pi_1(M_1))_2^{\mathbf{Q}} \cong \mathbf{Z}^{\beta_1(M_0)}$. $\square$

Example 2.6: It is important to note that "N-surgery related" is *not* a symmetric relation if $N = \{e\}$ or $N = G''$. In a later paper we will consider strengthening the N-surgery relation to *force* symmetry. Figure 2.7a shows a "Kirby calculus" description of $M_0 = S^1 \times S^2$ with a dashed curve γ which is clearly null-homotopic in M_0. Yet $+1$ surgery

FIGURE 2.7

along γ yields M_1 which (since the Whitehead link is symmetric) is homeomorphic to the manifolds of Figure 2.7b and 2.7c. Hence M_1 is 0-surgery on a left-handed trefoil knot and the loop γ' is neither null-homotopic in M_1 nor even in $\big(\pi_1(M_1)\big)''$. If M_0 were N-surgery related to M_1 for $N = G''$ (or $\{e\}$) then by 2.3 there would exist an epimorphism from $\mathbf{Z} = \pi_1(M_0)$ to $\pi_1(M_1)/N\big(\pi_1(M_1)\big)$. Since the trefoil knot has non-trivial Alexander module, this is not possible. One also sees that, in the case $N = \{e\}$, forcing symmetry would force π_1 itself to be preserved (since 3-manifold groups are Hopfian) and *perhaps* this is too strong to be of interest.

We have defined our "equivalence" relations to be generated by single Dehn surgeries. It is also possible to define relations generated by surgeries on certain types of *links*. These are sometimes equivalent notions as the following show. The proof of the first is elementary and left to the reader.

Proposition 2.8. *The following are equivalent.*

1. *M_0 and M_1 are 2-surgery equivalent.*
2. *There is a link $L = \{L_1, \ldots, L_m\}$ in M_0 with null-homologous components, each framed ± 1, with $\ell k(L_i, L_j) = 0$ so that M_1 is the result of surgery along L. In other words, the "linking matrix" of L is invertible over $\mathbf{Z}$ and diagonal.*

Proposition 2.9. *The following are equivalent.*

1. *M_0 and M_1 are rationally 2-surgery equivalent.*
2. *There is a framed link $L = \{L_1, \ldots, L_m\}$ in M_0, each component of which is*

rationally null-homologous in M_0, such that the "linking matrix" of L is non-singular over $\mathbf{Q}$ and such that M_1 is obtained by surgery on L.

Here, by "linking matrix" of the framed link $L = \{\gamma_1, \ldots, \gamma_m\}$ we mean the matrix over $\mathbf{Q}$ given by $v_{ij} = \ell k(\rho_i, \gamma_j)$ where ρ_i here is the circle on $\partial N(\gamma_i)$ which bounds the meridional disk in the surgery solid torus ($p_i\mu_i + q_i\rho_i$ in the notation of §1). The proof of 2.9 is given after the proof of Theorem 4.2.

§**3. Integral Homology Surgery Equivalence of Closed 3-manifolds.** In this chapter we give a comprehensive treatment of 2-surgery equivalence of closed 3-manifolds. This is precisely the equivalence relation generated by Dehn surgeries which preserve integral homology and thus was called *HTS-equivalence* (homologically trivial surgery) by Kuperberg [Ku] and Gerges [Ge]. This equivalence relation is perhaps the most basic and important. It forms the basis of the philosophy of Cochran and P. Melvin in their theory of finite type invariants for arbitrary 3-manifolds [CM]. The question of characterizing 2-equivalence was asked by Kuperberg and answered by Gerges in his Ph.D. thesis. Here we prove a sharper theorem. Our characterization theorem says that M_0 and M_1 are HTS equivalent precisely when they have the same H_1 and some isomorphism induces an isomorphism of $\mathbf{Q}/\mathbf{Z}$ linking forms and that part of the cohomology ring coming from triple cup products. In the next chapter we will prove the characterization theorem, which appears in Gerges [Ge] without the relation to the linking form. In this chapter we will discuss examples, invariants and representatives for the 2-equivalence classes.

Before stating the theorem, we set up some notation. We let $K(H_1(M_0), 1)$ be the usual Eilenberg-Maclane space with fundamental group $H_1(M_0)$. We can build this space from M_0 by adding cells of dimension greater than 1 and we let $f_0 : M_0 \to K(H_1(M_0), 1)$ denote this inclusion. Then if $\phi_1 : H_1(M_1) \to H_1(M_0)$ is any isomorphism, there is a unique homotopy class $f_1 : M_1 \to K(H_1(M_0), 1)$ inducing ϕ_1 on H_1. Let $B : H^1(\_; \mathbf{Z}_n) \to H^2(\_; \mathbf{Z})$ denote the Bockstein operator associated with the short exact sequence $0 \longrightarrow$

$\mathbf{Z} \xrightarrow{n} \mathbf{Z} \xrightarrow{\tau} \mathbf{Z}_n \longrightarrow 0.$

The following theorem in the case of homology 3-spheres was certainly known to and used by Andrew Casson in public lectures at M.S.R.I. in 1985. We are informed that it was known even earlier. In this case it says merely that any two oriented homology 3-spheres are 2-surgery equivalent. This case also appeared in a 1987 paper of S.V. Matveev. In the latter, moreover, it is proved that two 3-manifolds have isomorphic H_1 and linking forms if and only if they are related by "Borromean surgeries," a result clearly close in spirit to our final one [Ma, Theorem 2 and Remark 2].

The equivalence of B and (a slightly stronger version of) D is claimed in passing in [Tu1], but no proof is offered.

Theorem 3.1. (see [Ge]) *Suppose M_0 and M_1 are closed, oriented, connected 3-manifolds. The following 4 conditions are equivalent.*

A) *M_0 and M_1 are 2-surgery equivalent, i.e., each can be obtained from the other by a sequence of ± 1 surgeries (equivalently $\pm 1/q$ surgeries) or null-homologous circles.*

B) *There exists an isomorphism $\phi_1 : H_1(M_1) \to H_1(M_0)$ such that $(f_0)_*([M_0]) = (f_1)_*([M_1])$ in $H_3(H_1(M_0); \mathbf{Z})$ where f_0, f_1 are as above, In brief one could also say that M_0 and M_1 have the same homology and are bordant over $K(H_1(M_0), 1)$ for some f_i.*

C) *There exists an isomorphism $\phi_1 : H_1(M_1) \to H_1(M_0)$ such that the set of induced maps $\phi_n^1 : H^1(M_0; \mathbf{Z}/n\mathbf{Z}) \to H^1(M_1; \mathbf{Z}/n\mathbf{Z})$ for $n = 0$ and for each $n = p^r$ where p^r is the exponent of the p-torsion subgroup of $H_1(M_0; \mathbf{Z})$ (all elements of order some power of the prime p) satisfies the following:*

a) $\langle \alpha \cup \beta \cup \gamma, [M_0] \rangle = \left\langle \phi_n^1(\alpha) \cup \phi_n^1(\beta) \cup \phi^1(\gamma), [M_1] \right\rangle$ *where α, β, $\gamma \in H^1(M_0; \mathbf{Z}/n\mathbf{Z})$ and $[M_i]$ denotes the fundamental class in $H_3(M_i; \mathbf{Z}/n\mathbf{Z})$,*

b) $\langle \alpha \cup \tau_* B(\gamma), [M_0] \rangle = \left\langle \phi_n^1(\alpha) \cup \tau_* B\left(\phi_n^1(\gamma)\right), [M_1] \right\rangle$ *where α, γ, B are as above, but $n \neq 0$, and $\tau : H^2(M_i; \mathbf{Z}) \to H^2(M_i; \mathbf{Z}_n)$.*

D) *The same condition as* C *with* b) *replaced by*

c) *If* λ_i *represent the* $\mathbf{Q}/\mathbf{Z}$ *linking forms on* $T(H_1(M_i))$ *then* $\lambda_1(x,y) = \lambda_0(\phi_1(x), \phi_1(y))$ *for all* x, $y \in T(H_1(M_1))$, *that is to say that* ϕ_1 *induces an isomorphism between* λ_0 *and* λ_1.

Let A be a finitely generated abelian group and $A_n^* = \text{Hom}(A; \mathbf{Z}_n) \equiv H^1(A; \mathbf{Z}_n)$ for $n = 0$ or $n = p^r$ (the exponent of the p-torsion subgroup of A). Consider a set of skew-symmetric trilinear forms $u_n : A_n^* \times A_n^* \times A_n^* \longrightarrow \mathbf{Z}_n$, where n ranges over $\{0, p^r\}$ as above, which are compatible in the sense of [Tu2; Definition 1.2]. Let λ : Torsion $A \times$ Torsion $A \longrightarrow \mathbf{Q}/BZ$ be a non-degenerate symmetric bilinear form. Any automorphism $\phi : A \to A$ induces isomorphic forms $\{\phi^*(u_n)\}$ and $\{\phi_* \lambda\}$ given by $\phi^*(u_n)(\alpha, \beta, \gamma) = u_n(\phi_n^*(\alpha), \phi_n^*(\beta), \phi_n^*(\gamma))$ where $\phi_n^* : A_n^* \to A_n^*$ and $\phi_*\lambda(x,y) = \lambda(\phi^{-1}x, \phi^{-1}y)$. Given any oriented 3-manifold with $H_1 \cong A$, the triple cup product forms and linking form yield a pair $(\{u_n\}, \lambda)$ which is well-defined up to isomorphism. Let $\mathcal{S}(A)$ be the set of isomorphism classes of such pairs which are realizable by a 3-manifold. In fact by [Tu2; Theorem 1] and [KK], *any* pair is realizable if A has no 2-torsion. In general there is a mild compatibility condition between u_{2^r} and λ. Hence Theorem 3.1 may be restated as follows.

Theorem 3.1 (Restatement). *Let* A *be a finitely generated abelian group. The set of surgery equivalence classes of closed, oriented* 3*-manifolds with* $H_1 \cong A$ *is in bijection with* $\mathcal{S}(A)$.

Example 3.2 ($H_1 \cong \mathbf{Z}^m$ $m < 3$)**:** Any two homology 3-spheres are surgery equivalent since B is trivially satisfied in this case. Indeed since $H_3(\mathbf{Z}^m) = 0$ if $m < 3$, any two 3-manifolds M_0, M_1 with $H_1 \cong \mathbf{Z}^m$ are surgery equivalent if $m < 3$.

Example 3.3 $H_1 \cong \mathbf{Z}^3$**:** If $M_0 = \#_{i=1}^3 S^1 \times S^2$ and $M_1 = S^1 \times S^1 \times S^1$ then M_0 is not surgery equivalent to M_1 because the image of M_0 in $H_3\big(H_1(M_0)\big)$ is zero since

it factors through $H_3\big(\pi_1(M_0)\big)$. But for *any* automorphism of $\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}$, the induced map $M_1 \to S^1\times S^1\times S^1 = K\big(H_1(M_0),1\big)$ is of degree ±1 since the identity map is degree 1 and $\operatorname{Aut}\big(H_3(\mathbf{Z}^3)\big) = \{\pm1\}$. Hence condition B fails (equivalently condition C part a with $n=0$). More generally, let M_n be the 3-manifold shown in figure 3.4 as zero surgery on a link with $\overline{\mu}(123) = n$. Then M_n is surgery equivalent to M_m if and only if $|n| = |m|$ since the triple cup product of the Hom-duals of the meridians is n times the fundamental class in H^3. Any element of $\operatorname{Aut}(\mathbf{Z}^3)$ induces an element of $\operatorname{Aut}\big(H_3(\mathbf{Z}^3)\big)$ and this correspondence is $P \to \det P$. Since $\det P = \pm1$, the classes n, m in $H_3(\mathbf{Z}^3)$ are equivalent under the action of $\mathrm{GL}(3,\mathbf{Z})$ if and only if $n = \pm m$. Note that this implies that M_0 and M_1 are surgery equivalent if and only if they have identical Lescop invariant [Les]. This generalizes to give the following.

FIGURE 3.4

Corollary 3.5. *Let $\mathcal{S}_m$ be the set of surgery equivalence classes of closed oriented 3-manifolds with $H_1 \cong \mathbf{Z}^m$. Then there is a bijection $\psi_* : \mathcal{S}_m \to \Lambda^3(\mathbf{Z}^m)/\,\mathrm{GL}_m(\mathbf{Z})$ from $\mathcal{S}_m$ to the set of orbits of the third exterior power of $\mathbf{Z}^m$ under the action induced by $\mathrm{GL}_m(\mathbf{Z})$ on $\mathbf{Z}^m$. Any such manifold is surgery equivalent to one which is the result of 0-framed surgery on a m component link in S^3 which is obtained from the trivial link by replacing a number of trivial 3-string braids by a 3-string braid whose closure is the Borromean rings.*

Proof of 3.5. Fix an identification $H_3\big((S^1)^m\big) \equiv H_3(\mathbf{Z}^m) \equiv \Lambda^3\mathbf{Z}^m$. For any M with $H_1 \cong \mathbf{Z}^m$ choose an isomorphism $\phi : H_1(M) \to \mathbf{Z}^m$. This induces a unique homotopy

class of maps $\psi : M \to (S^1)^m$. The image of the fundamental class of M is the desired element $\psi_*([M])$. All possible isomorphisms ϕ may be achieved by post-composing a fixed ϕ with elements of $\mathrm{GL}_m(\mathbf{Z})$. Then $\psi_*(M)$ is well-defined in the orbit space. If M and M' are surgery equivalent then condition B of 3.1 guarantees their images are the same. Hence ψ_* is well-defined. $B \Rightarrow A$ implies that ψ_* is injective. The surjectivity of ψ_* follows from work of D. Sullivan [Su]. Alternatively, for any set of $\binom{m}{3}$ integers $\{a_{ijk} \mid 1 \leq i < j < k \leq m\}$ is it easy to construct an ordered link of m components in S^3 such that $\overline{\mu}(ijk) = a_{ijk}$ by the procedure described in the last sentence of the Corollary. If M is zero surgery on this link then $\psi_*(M) = \sum a_{ijk}(e_i \wedge e_j \wedge e_k)$ with respect to a basis induced by the meridians (see Lemma 4.2 of [Tu2]). $\square$

The structure of the set $\Lambda^3(\mathbf{Z}^m)/\,\mathrm{GL}_m$ seems to be quite complicated for large m and so the general decidability question for whether or not two 3-manifolds are surgery equivalent may not be easy. However since the 0 element is the only element in its orbit we have:

Corollary 3.6. *A 3-manifold with $H_1 \cong \mathbf{Z}^m$ is surgery equivalent to $\#_{i=1}^m S^1 \times S^2$ if and only if its integral triple cup product form $H^1 \oplus H^1 \oplus H^1 \to \mathbf{Z}$ vanishes identically.*

We also observe the following surprising result.

Corollary 3.7. *The map $\mathcal{S}_3 \xrightarrow{f} \mathcal{S}_4$ given by $M \to M\#S^1 \times S^2$ is a bijection. Thus any N with $H_1(N) \cong \mathbf{Z}^4$ is surgery equivalent to precisely one of $M_n\#S^1 \times S^2$ where $n \geq 0$ (see Figure 3.4).*

Proof of 3.7. $\Lambda^3\mathbf{Z}^4 \cong \Lambda^1\mathbf{Z}^4 \cong \mathbf{Z}^4$ by duality. Thus $\Lambda^3\mathbf{Z}^4/\,\mathrm{GL}_4 \cong \mathbf{Z}^4/\,\mathrm{GL}_4 \cong \mathbf{Z}_+ \cup \{0\}$. Under this bijection, $n(e_1 \wedge e_2 \wedge e_3)$ goes to ne_4 and the former is $\psi_*(M_n\#S^1 \times S^2)$. $\square$

Corollary 3.8. *If $H_1(M)$ is torsion-free then M is surgery equivalent to $-M$.*

Proof of 3.8. The element of GL_m which reverses the order of a basis $\{e_1, \ldots, e_m\} \to \{e_m, \ldots, e_1\}$ induces -1 on $\Lambda^3\mathbf{Z}^m$. $\square$

Now consider that $H_1 \cong \mathbf{Z}_n$. First consider the general question of which classes $\mu \in H_3(A)$ can be realized as the image of the fundamental class of a 3-manifold M_0 under some map $\pi_1(M_0) \twoheadrightarrow H_1(M_0) \xrightarrow{f} A$ where f is an *isomorphism*. This question has been answered by Turaev in great generality. The answer is that μ is realizable if and only if $x \mapsto x \cap \mu$ is an isomorphism $\operatorname{Tors} H^2(A) \to \operatorname{Tors} H_1(A)$. Both groups are $\mathbf{Z}_n$ in the case at hand. If we denote by $\mu = 1 \in H_3(\mathbf{Z}_n) = \mathbf{Z}_n$ the image of the class of $L(n,1)$ under some map then certainly $x \mapsto x \cap 1$ is an isomorphism. A general class $\mu = k \cdot 1$ will induce the map $x \mapsto k(x \cap 1)$ which is the composition of the isomorphism $x \mapsto x \cap 1$ with multiplication by k on $\mathbf{Z}_n$. Hence $k \in H_3(\mathbf{Z}_n)$ is realizable if and only if k is a unit in $\mathbf{Z}_n$. Moreover by 3.1 B, two 3-manifolds M_0, M_1 with $H_1 \cong \mathbf{Z}_n$ representing classes k_0, k_1 in $H_3(\mathbf{Z}_n)$ with respect to *some* identifications $H_1(M_i) \cong \mathbf{Z}_n$, will be surgery equivalent if and only if there is an automorphism of $\mathbf{Z}_n$ which induces an automorphism of $H_3(\mathbf{Z}_n)$ sending k to k_1. Multiplication by (a unit) m on $\mathbf{Z}_n$ induces multiplication by m^2 on $H_3(\mathbf{Z}_n)$ (see Proposition 3 of [Ru] and Theorem 29.5 of [Co]). Therefore we have derived the following.

Corollary 3.9. *The set $\mathcal{S}(\mathbf{Z}_n)$ of surgery equivalence classes of closed oriented 3-manifolds with $H_1 \cong \mathbf{Z}_n$ is in bijection with the set of equivalence classes of units of $\mathbf{Z}_n$, modulo squares of units, the correspondence being given by the image of the fundamental class of the manifold in $H_3(\mathbf{Z}_n) \equiv \mathbf{Z}_n$. The correspondence is also given by the self-linking linking number of a generator of H_1 ($\lambda(1,1) = q$ and q is viewed as an element of $\mathbf{Z}_n$). The equivalence class of $q \in \mathbf{Z}_n$ contains the lens space $L(n,q)$.*

Proof of 3.9. The first statement is proved above using 3.1 B. For the second statement, use 3.1 D. Note that since H_1 is cyclic, the cup products $H^1 \oplus H^1 \to H^2$ must vanish by anticommutativity with any coefficients unless n is even. Therefore if n is odd, part a) of 3.1 D is vacuous. For even n the triple cup product form on a cyclic group is determined by the linking form [Tu2; Theorem 1], so in any case we need only consider condition c)

of 3.1 D. Clearly the linking form λ on a cyclic group is determined by $\lambda(1,1) = \frac{a}{n}$ and $a \in \mathbf{Z}_n$ is well-defined modulo squares of units. Moreover two such forms λ and λ' are isomorphic if and only if $a \equiv a'$ modulo squares. For $L(n,q)$, $\lambda(1,1) = \frac{q}{n}$. $\square$

Corollary 3.10. *($H_1 \cong \mathbf{Z}_p$, p prime): Any 3-manifold with $H_1 \cong \mathbf{Z}_2$ is surgery equivalent to $\mathbf{R}P(3)$. For any odd prime there are precisely two surgery equivalence class represented by $L(p,1)$ and $L(p,q)$ where $q \not\equiv k^2 \mod p$. Hence if $p \equiv 3 \mod 4$ then we may take $q = -1$ and we see that $L(p,1)$ and $-L(p,1)$ are not surgery equivalent. If $p \equiv 1 \mod 4$ then $-L(p,1)$ is surgery equivalent to $L(p,1)$.*

More generally we see that:

Corollary 3.11. *If M_0 and M_1 are orientation-preserving homotopy equivalent then they are surgery equivalent.*

Proof of 3.11. This is immediate from 3.1 B.

Corollary 3.12. *If $\pi_1(M_0) \cong \pi_1(M_1)$ is abelian then M_0 is surgery equivalent to M_1 if and only if M_0 is orientation-preserving homotopy equivalent to M_1.*

Proof of 3.12. One implication follows from 3.11. Suppose M_1 is surgery equivalent to M_0. By Theorem 4.2 there is cobordism W from M_0 to M_1 built from $M_0 \times [0,1]$ by adding two-handles attached along curves being in $[\pi_1(M_0), \pi_1(M_0)] = 0$. Hence $W \simeq M_0 \vee S^2 \cdots \vee S^2$, and there is a retraction $r : W \to M_0$. The inclusion $M_1 \to W$ followed by r is a degree 1 map $M_1 \to M_0$ inducing an isomorphism on π_1 and all homology groups. We may assume the manifolds contain no fake 3-cells since these are irrelevant to the question of being homotopy equivalent. Since π_1 is abelian it is not a non-trivial free product so we may assume that $\pi_2(M_0) = \pi_2(M_1) = 0$ or that $M_0 \cong M_1 \cong S^1 \times S^2$. In the first case π_1 must be finite cyclic and then it is easy to see that f induces an isomorphism on π_3 by considering the universal cover of W. Hence f is a degree 1 homotopy equivalence. $\square$

Corollary 3.13. *$L(n,q)$ is surgery equivalent to $L(n,q')$ if and only if they are orientation-preserving homotopy equivalent, that is if $qq' \equiv k^2 \mod n$ for some unit k.*

Example 3.14 ($H_1 \cong \mathbf{Z} \times \mathbf{Z}_n$): Since $H_3(\mathbf{Z} \times \mathbf{Z}_n) \xrightarrow{\pi} H_3(\mathbf{Z}_n)$ is an isomorphism, the surgery equivalence class depends only on the linking form. From another point of view, since $H^1(\mathbf{Z} \times \mathbf{Z}_n; \mathbf{Z}_m)$ is generated by 2-elements, the triple cup product forms vanish if m is odd and are determined by the linking form if m is 2^r. Therefore each surgery equivalence class contains a representative of the form $S^1 \times S^2 \# L(n,q)$ and the self-linking number of an element of order n in $H_1(M)$, $\lambda(1,1) = \frac{a}{n}$ viewed as an element $a \in \mathbf{Z}_n$ will distinguish the classes when viewed in the group of units modulo squares (as in the case $H_1 \cong \mathbf{Z}_n$).

Example 3.15: Some words of caution are in order. One must be careful in applying 3.1. There exist 3-manifolds which have isomorphic H_1, linking forms and *integral* triple cup product forms but are not surgery equivalent as detected by a $\mathbf{Z}_p$ triple cup product form. Namely, let M_0 be $\#_{i=1}^3 L(5,1)$ and let M_1 be 5/1-surgery on each component of a Borromean Rings.

It is even possible that M_0 and M_1 have isomorphic linking forms and isomorphic triple cup product forms with all coefficients, yet *not* be surgery equivalent because the isomorphisms are not induced by the same isomorphism ϕ on H_1! Consider the manifolds in Figure 3.16. Since they have the same linking matrix (expand the 5/2 to a chain of integral surgeries if you like), their linking forms are isomorphic to $(1/5) \oplus (2/5)$ on $\mathbf{Z}_5 \times \mathbf{Z}_5$. The triple cup product form on integral H^1 is zero while the triple cup product forms on $H^1(\_; \mathbf{Z}_5) \equiv (\mathbf{Z}_5)^4$ are isomorphic by "swapping the meridians of the 5 and 5/2 knots." But these isomorphisms are incompatible. We do not provide details.

We can relate surgery equivalence to two other geometric equivalence relations which have appeared in the literature. The first is concerned with Dehn surgery on links; the

FIGURE 3.16

second is concerned with Heegard splittings and mapping class groups.

Recall that, given M_0, any closed, oriented 3-manifold M_1 can be obtained by integral surgery on *some* framed link in M_0. If the links are restricted what can be said? Recall that a *boundary link* is a very special link with all linking numbers zero, namely one whose components bound disjoint Seifert surfaces. This makes sense in any 3-manifold. The following is mildly surprising.

Proposition 3.17. *Let M_0, M_1 be oriented, connected 3-manifolds. Then the following are equivalent:*

A. *M_0 is 2-surgery equivalent to M_1*
B. *There is a framed boundary link L in M_0 such that Dehn surgery (with framings ± 1) on L yields M_1.*
C. *M_0 and M_1 have isomorphic linking forms and triple cup product forms (in the sense of 3.1).*

Once again the result for homology spheres, where C is vacuous, was known. The earliest reference we can find is S.V. Matveev [Ma; Theorem A]. This result was reproved and used by S. Garoufalidis [Ga].

Proof of 3.17. The equivalence of A and C is part of 3.1. B⇒A is almost immediate. One need only note that the remaining components of a boundary link remain null-homologous (use the same Seifert surface) after performing Dehn surgery on some

of its components. Similarly the framings remain ± 1 because the longitude remains the same. Thus we need only establish A$\Rightarrow$B. By 2.8 we can assume that M_1 is the result of ± 1 surgeries on an link in M_0 whose pairwise linking numbers are zero. By induction, suppose M_1 is ± 1 surgery on a null-homologous knot K in $\mathcal{S}(M_0, \{L_1, \ldots, L_n\})$, the result of ± 1-framed surgery on the boundary link L in M_0, where $\ell k(K, L_i) = 0$ for all i. The following type of argument has been used by others to prove the case of homology spheres. It serves equally well in general. Let $L' \subseteq \mathcal{S}$ be the link consisting of the cores of the surgery solid torii (so $\mathcal{S} - L' \equiv M_0 - L$). We shall describe an isotopy of K to K' in $\mathcal{S}$ (passing through L') such that $L \cup K'$ is a boundary link in M_0. Consider a set $\mathcal{W} = \{W_1, \ldots, W_n\}$ of disjoint Seifert surfaces for L in $M_0 - L$ whose boundaries are longitudes. Since we may actually assume the homeomorphisms defining the Dehn surgeries carry longitudes to longitudes, these surfaces can be extended by adding annuli to $\widehat{\mathcal{W}} = \{\widehat{W}_1, \ldots, \widehat{W}_n\}$, disjoint Seifert surfaces in $\mathcal{S}$ for the components of L'. Now K bounds a surface V in $M_0 - L \equiv \mathcal{S} - L'$ because of the hypothesis on linking numbers. Hence $\partial \widehat{W}_i \cap V = \phi$. V has a 1-dimensional spine whose transverse intersections with $\widehat{\mathcal{W}}$ may be removed by isotopy merely by pushing over $\partial \widehat{\mathcal{W}}$. This isotopy extends to V. The resulting $K' = \partial V'$ forms a boundary link with L since $V' \cap \mathcal{W} \subseteq V' \cap \widehat{\mathcal{W}} = \phi$. Note that, in the presence of other components K_2, $K_3, \ldots$ such that $\ell k(K, K_i) = 0$, the isotopy can be chosen so as to preserve that latter fact. $\square$

The reader might find it interesting to compare this with Theorem B of [Ma] which maintains that M_0 and M_1 have isomorphic homology groups and linking forms if and only if M_1 can be obtained from M_0 by surgery on a "T_0-boundary link" (recently re-introduced by Garoufalidis and Levine who used the term "blink" [GL]). This was an improvement on a theorem of Hilden who showed that any homology 3-sphere can be obtained from S^3 by surgery on a blink [H].

Another way to describe 3-manifolds is by their Heegard splittings. Given M_0, by

choosing a Heegard splitting $M_0 = H_1 \cup_f H_2$, one can vary f by composing with another homeomorphism g and in doing so change the three-manifold. One can then ask if an arbitrary M_1 may be obtained in such a manner (it *can* by Lickorish's theorem), or if it can be obtained using only g taken from some subgroup of the group of homeomorphisms. In particular, if Γ is the mapping class group for the closed, orientable surface of genus g then we can consider the *Torelli group* $\mathcal{T}$, and the *Johnson group* $\mathcal{K}$ of Γ. The Torelli group is the subgroup generated by homeomorphisms inducing the identity map on H_1. The *Johnson subgroup* $\mathcal{K} \subseteq \mathcal{T}$ is the subgroup of Γ generated by Dehn twists along simple closed curves in the surface which bound a subsurface [Jo1] [Mo1] [Mo2] [GL]. Then, building on the observations of [GL] and our own work we have the following generalization of the theorem of Morita for homology spheres [Mo1; Prop. 2.3].

Theorem 3.18. *Let M_0, M_1 be closed oriented 3-manifolds. Then the following are equivalent.*

A. *M_0 and M_1 are 2-surgery equivalent.*

B. *There exist Heegard splittings $M_0 = H_1 \cup_f H_2$, $M_1 = H_1 \cup_{\psi \circ f} H_2$ where $\psi \in \mathcal{K}$.*

C. *M_0 and M_1 have isomorphic linking forms and triple cup product forms (as in 3.1).*

Proof of 3.18. The arguments in the first 3 paragraphs of section 2.3 of [GL], although given for homology spheres, suffice to show that 3.18B is equivalent to 3.17B. □

Remark 3.19: We have shown that $\mathcal{K}$ corresponds to boundary links which, in turn, corresponds to homology surgery equivalence. Strangely, the seemingly more natural Torelli group corresponds to "blinks" (see [Ma] and [GL]) which corresponds to preserving only H_1 and the linking form. Combining the work of [Ma], [GL] and our present work yields an analagous theorem to this effect, with "homology surgery equivalence" changed so that the relation is generated by surgery on a 2-component blink rather than a knot.

§4. **Proofs of Theorem 3.1 and other Basic Theorems.** In this section we prove Theorem 3.1. Several major components of the proof are derived in much greater generality so that they can be employed in later sections.

A $\Rightarrow$ B: We will prove a more general result which will be useful later.

Proposition 4.1. *Suppose M_1 is obtained from M_0 by ± 1 surgeries on a link $\{\gamma_1, \ldots, \gamma_n\}$ where $\gamma_i \in N\big(\pi_1(M_0)\big)$ and the meridians $\gamma_i' \in N\big(\pi_1(M_1)\big)$. Then there exists an isomorphism $\phi : \pi_1(M_1)/N\big(\pi_1(M_1)\big) \to \pi_1(M_0)/N\big(\pi_1(M_0)\big)$ such that $(f_0)_*([M_0]) = (f_1)_*([M_1])$ in $H_3\left(\pi_1(M_0)/N\big(\pi_1(M_0)\big); \mathbf{Z}\right)$ where f_1 induces ϕ on π_1 and f_0 is the "natural inclusion." Equivalently, there exists an f_1, inducing an isomorphism ϕ, such that (M_0, f_0) and (M_1, f_1) are bordant over $K\left(\pi_1(M_0)/N\big(\pi_1(M_0)\big), 1\right)$.*

Proof of 4.1. Let W be the usual cobordism from M_0 to M_1 obtained by attaching 2-handles to the γ_i in $M_0 \times \{1\} \subset M_0 \times [0,1]$. The curves γ_i' are the attaching circles of the dual 2-handles attached to M_1. Hence the inclusions j_0, j_1 induce isomorphisms $(j_0)_*$, $(j_1)_*$ on π_1 "modulo the N-subgroup." Let $\phi = (j_0)_*^{-1} \circ (j_1)_*$. Let $\psi : \pi_1(W)/N\big(\pi_1(W)\big) \to \pi_1(M_0)/N\big(\pi_1(M_0)\big)$ be $(j_0)_*^{-1}$. Then there are continuous maps f_0, F and f_1 from M_0, W, M_1 respectively inducing the projection, ψ and ϕ respectively on π_1 and such that F extends f_i. The result follows. $\square$

To see that 4.1 implies $A \Rightarrow B$, we apply 2.3 to reduce to the case of ± 1 surgeries, we note that it suffices to prove $A \Rightarrow B$ for the case of a single ± 1 surgery, then appeal to 2.1 to see that the hypotheses of 4.1 are satisfies.

A $\Rightarrow$ D: It suffices to consider the case that M_0 and M_1 are cobordant via a single 2-handle attached with ± 1 framing along a circle γ which is null-homologous in M_0. Let $\phi_1 = (j_0)_*^{-1} \circ (j_1)_*$ be the isomorphism on H_1 induced by the inclusions. Then ϕ_1 algebraically induces ϕ_n^1 as in C and the key point is to observe that in this case ϕ_n^1 equals

$j_1^* \circ (j_0^*)^{-1}$ where j_i^* are the isomorphisms on $H^1(\_;\mathbf{Z}_n)$ induced by the inclusions. Hence for any α, β, γ in $H^1(M_0;\mathbf{Z}_n)$, we have

$$\langle \phi_n^1(\alpha) \cup \phi_n^1(\beta) \cup \phi_n^1(\gamma), [M_1] \rangle = \langle (j_0^*)^{-1}(\alpha) \cup (j_0^*)^{-1}(\beta) \cup (j_0^*)^{-1}(\gamma), (j_1)_*([M_1]) \rangle$$

since j_1 is a continuous map. But $(j_0)_*([M_0]) = (j_1)_*([M_1]$ for any coefficients so the above expression equals $\langle \alpha \cup \beta \cup \gamma, [M_0] \rangle$ as desired. This shows condition a) of D and C. Now we demonstrate that $\lambda_1(\phi_1^{-1}x, \phi_1^{-1}y) = \lambda_0(x,y)$ for all torsion classes x, y in $H_1(M_0;\mathbf{Z})$. We may choose circles $\tilde{x}$, $\tilde{y}$ in M_0 to represent these classes so that they are disjoint from γ and are in fact disjoint from a Seifert surface S for $\ell(\gamma)$. This is true because if $\tilde{x}$ hits S, we are free to isotope $\tilde{x}$ "through γ" to achieve that the algebraic number of such intersections is 0. Then modify S to miss $\tilde{x}$. Choose an integer n and a 2-chain d of M_0 such that $\partial d = n\tilde{x}$ and such that d meets $\tilde{y}$ and γ transversely. Then $\lambda_0(x,y)$ equals $\frac{1}{n}$ times $\#(\tilde{y} \cdot d)$. Now note that $\tilde{x}$ and $\tilde{y}$ are perfectly good representatives of $\phi_1^{-1}(x)$ and $\phi_1^{-1}(y)$ since ϕ_1 is induced by the inclusions. We construct d', a 2-chain in M_1 such that $\partial d' = \partial d = nx'$. For each intersection of d with γ, delete the 2-disk $d \cap N(\gamma)$ and replace it by a copy of the annulus in the surgery solid torus which expresses the fact that $\mu(\gamma)$ is isotopic to $\pm\ell(\gamma)$ after surgery, and a copy of $\pm S$. This 2-chain d' lies in M_1, has the same boundary as d and $d' \cdot \tilde{y} = d \cdot \tilde{y}$ since $\tilde{y} \subseteq M_0 - N(\gamma)$ and $\tilde{y}$ is disjoint from S. Hence $\lambda_1\big(\phi_1^{-1}(x), \phi_1^{-1}(y)\big) = \lambda_0(x,y)$. $\square$

B $\Rightarrow$ A: We will prove a significantly broader result than is necessary in order to use it in later sections.

Theorem 4.2. *Suppose M_0 and M_1 are closed, oriented 3-manifolds. Suppose $N \trianglelefteq \pi_1(M_0)$ is the normal closure of a finite number of elements and that N is contained in the commutator subgroup of $\pi_1(M_0)$ (or merely in $\big(\pi_1(M_0)\big)_2^{\mathbf{Q}}$ in the rational case). Suppose there exists an epimorphism $\phi : \pi_1(M_1) \to \pi_1(M_0)/N$ which induces an isomorphism on $H_1(\_;\mathbf{Z})$ (or merely $H_1(\_;\mathbf{Q})$ in the rational case) such that $(f_0)_*([M_0]) = (f_1)_*([M_0])$ in*

$H_3\big(\pi_1(M_0)/N;\mathbf{Z}\big)$ *(here* f_1 *is induced by* ϕ *as usual, and* f_0 *induced by the inclusion into* $K\big(\pi_1(M_0)/N,1\big)$*. Then* $(M_0,f_0)=(M_1,f_1)$ *in* $\Omega_3\big(K(\pi_1(M_0)/N,1)\big)$ *via a* 4*-manifold with only* 2*-handles* $(\operatorname{rel} M_0)$ *whose attaching circles lie in* N *and whose linking matrix with respect to* M_0 *is diagonal and invertible over* $\mathbf{Z}$ *(merely invertible over* $\mathbf{Q}$ *in the rational case). Consequently* M_0 *is obtained from* M_1 *by* ± 1 *surgeries (integral surgeries in the rational case) on a link* $\{\gamma_1,\ldots,\gamma_n\}$ *such that* $[\gamma_i]\in N$*. This link may be ordered in such a way that the sequence of these surgeries exhibits that* M_1 *is* N*-surgery related to* M_0*. Conversely* M_1 *is obtainable from* M_0 *in a similar manner by surgery on a link* $\{\gamma_i'\}$ *and that* $\ker\phi$ *is the normal subgroup of* $\pi_1(M_1)$ *generated by* $\{\gamma_i'\}$*. Example 2.6 shows that this cannot, in general, be strengthened.*

Before proving 4.2, we should check that it implies $B \Rightarrow A$. Apply 4.2 with $N=\big(\pi_1(M_0)\big)_2$ to get the link $\{\gamma_1,\ldots,\gamma_n\}$. Apply 2.8. Hence M_1 is 2-equivalent via integral surgeries to M_0.

Proof of 4.2. Let $X=K\big(\pi_1(M_0)/N,1\big)$ which we may think of as constructed by adjoining cells to M_0. Then we have natural maps $f_0: M_0\to X$ and $f_1: M_1\to X$ such that $(f_1)_*=\phi$. It is well known that the map from $\Omega_3(X)\to H_3(X)$ is an isomorphism given by the image of the fundamental class. Thus the hypotheses guarantee that there is a compact oriented 4-manifold W and a map $F: W\to X$ such that $\partial(W,F)=(M_1,f_1)\amalg(-M_0,f_0)$. Since F_* is necessarily surjective on π_1, we may perform surgery on circles in W and assume F_* is an isomorphism.

Choose a handlebody structure of $W \operatorname{rel} M_0$ with no handles of index 0 or 4. We may then proceed to "trade" 1-handles for 2-handles as in [Ki2, pp. 6–7, p. 247]. This may also be thought of as performing a surgery on the interior of W along a circle c passing over the 1-handle. Since $(f_0)_*$ is an epimorphism on π_1, these circles may be altered by loops in M_0 so that $F_*(c)=0$ (then in fact these loops are null-homotopic) and hence the map F extends to the "new" W. Since $\phi_*=(f_1)_*$ is surjective we may trade all 3-handles for

2-handles, by viewing then as 1-handles attached to M_1.

Now let V be the "linking matrix" of the attaching maps of the 2-handles rel M_0. By this we mean the following. If $\{\gamma_1, \ldots, \gamma_m\}$ denote the attaching circles and $\{\rho_1, \ldots, \rho_m\}$ are the surgery circles on $\partial N(\gamma_i)$ which are null-homotopic in M_1, then let $v_{ij} = \ell k(\rho_i, \gamma_j)$. Since ρ_i, γ_j are disjoint oriented circles in M_0 which are null-homologous (torsion in the $\mathbf{Q}$-case) v_{ij} is a well-defined integer (rational number in the $\mathbf{Q}$-case). We shall show that V is invertible over $\mathbf{Z}$ (respectively over $\mathbf{Q}$).

We treat the $\mathbf{Q}$ case first. Consider the long exact sequence in rational homology for the pair $(M_0, M_0 - \dot{N}(L))$ where $L = \{\gamma_1, \ldots, \gamma_m\}$.

$$\longrightarrow H_2\left(M_0, M_0 - \dot{N}(L)\right) \xrightarrow{\partial_*} H_1\left(M_0 - \dot{N}(L)\right) \xrightarrow{\pi_0} H_1(M_0) \longrightarrow 0.$$

Since the first term is $\mathbf{Q}^m$ generated by the meridional disks we get $\mathbf{Q}^m \xrightarrow{i} H_1\big(M_0 - \dot{N}(L)\big) \xrightarrow{\pi_0} H_1(M_0) \longrightarrow 0$ where $i(e_i) = \mu_i = \mu(\gamma_i)$. It is well-known that i is injective when γ_i are zero in $H_1(\_;\mathbf{Q})$. A splitting of i is given by $\phi(x) = \sum_{j=1}^m \ell k(x, \gamma_j) e_j$. Hence $\psi : H_1(M - L) \to Q^m \oplus H_1(M_0)$ by $x \to \big(\phi(x), \pi_0(x)\big)$ is an isomorphism. Note that $\psi(\rho_i) = \left(\sum_{j=1}^m \ell k(\rho_i, \gamma_j) e_i, 0\right)$ since $\rho_i = 0$ in $H_1(M_0;\mathbf{Q})$. Since $H_1(M_1) \cong H_1(M_0 - L)/\langle \rho_i \rangle$, $H_1(M_1) \cong H_1(M_0) \oplus \mathbf{Q}^{m - \operatorname{rank} V}$. But ϕ is an isomorphism on $H_1(\_;\mathbf{Q})$ so $\operatorname{rank} V = m$.

In the integral case we have the exact sequence in integral homology:

$$\mathbf{Z}^m \xrightarrow{i} H_1\left(M_0 - \dot{N}(L)\right) \xrightarrow{\pi_0} H_1(M_0) \longrightarrow 0$$

where i is injective because it is injective with rational coefficients. Here $i(e_i) = \mu_i$ and there is an isomorphism $\phi : \ker \pi_0 \to \mathbf{Z}^m$ given by $\phi(x) = \sum_{j=1}^m \ell k(x, \gamma_j) e_j$ such that $\phi \circ i = \text{identity}$. In particular $\phi(\rho_i) = \sum_{j=1}^m \ell k(\rho_i, \gamma_j) e_j = \sum_{j=1}^m v_{ij} e_j$ so $i\left(\sum_{j=1}^m v_{ij} e_j\right) = \rho_i$. Since $H_1(M_1;\mathbf{Z}) \cong H_1\big(M_0 - \dot{N}(L)\big)/\langle \rho_i \rangle$, we see that the cokernel of $\mathbf{Z}^m \xrightarrow{V} \mathbf{Z}^m$ embeds in $H_1(M_1;\mathbf{Z})$ via the map i. If this cokernel is non-zero then there is a class

$x \in \ker \pi_0$ which is *non-zero* under $\pi_1 : H_1\big(M_0 - \dot{N}(L)\big) \longrightarrow H_1(M_1)$. This implies that $\pi_1(x)$ is a non-trivial element in the kernel of the inclusion map $H_1(M_1) \longrightarrow H_1(W)$ which is a contradiction. Hence V is invertible over $\mathbf{Z}$.

Now that we have established that the linking matrix is invertible, in the integral case, we appeal to the classification of symmetric bilinear forms. We can change W by adding a single ± 1 framed 2-handle attached along a trivial circle in order to assume V is an indefinite, odd form. Such a form has a $\mathbf{Z}$-basis for which V is diagonal with ± 1's on the diagonal. This basis change can be realized geometrically by handle slides (see [Ki2; Chapter 2]). Thus M_1 is obtained from M_0 by ± 1 surgeries on a link $\{\gamma_1, \ldots, \gamma_n\}$ such that each $[\gamma_i] \in N\big(\pi_1(M_0)\big)$ and $\ell k(\gamma_i, \gamma_j) = 0$. Conversely M_0 is obtainable from ± 1 surgery on the dual link $\{\gamma_1', \ldots, \gamma_n'\}$ where, in general, all we know is that $[\gamma_i'] \in \operatorname{kernel} \phi$ which in turn lies in $[\pi_1(M_1), \pi_1(M_1)]$.

For the rational case we need the following Lemma, which may be well known. The proof was suggested to me by Richard Stong.

Lemma 4.3. *If $q : \mathbf{Z}^n \times \mathbf{Z}^n \longrightarrow \mathbf{Q}$ is a symmetric, bilinear, non-singular form then there is a basis $e_1, \ldots, e_n$ for $\mathbf{Z}^n$ such that q restricted to $\langle e_1, \ldots, e_i \rangle$ is non-singular for each $i \leq i \leq n$.*

Proof. First we claim that for any such form there is a basis such that $q(e_1, e_1) \neq 0$. For an arbitrary basis $\{e_1, \ldots, e_n\}$, if some j has $q(e_j, e_j) \neq 0$ then we are done by re-ordering. If all $q(e_j, e_j) = 0$ then, by non-singularity, there is some j such that $q(e_1, e_j) \neq 0$. Then the basis $\{e_1 + e_j, e_2, \ldots, e_n\}$ works.

Now we proceed by induction. Suppose we have a basis $\{e_1, \ldots, e_n\}$ such that $q| \langle e_1, \ldots, e_j \rangle$ is non-singular for each $j < i$. We shall re-choose $\{e_i, \ldots, e_n\}$ such that $q| \langle e_1, \ldots, e_i \rangle$ is also non-singular. To do so write q in our basis as $q = \begin{pmatrix} A & B \\ B' & C \end{pmatrix}$ where A is $(k-1)$ by $(k-1)$ and B' is the transpose of B. We can make a *rational* change of basis to replace

this matrix by $q^\wedge = \begin{pmatrix} A & O \\ O & D \end{pmatrix}$ where $D = C - B'A^{-1}B$. Since $q^\wedge$ is non-singular (since q is), D is a non-singular matrix. As in the first step of our proof, there is an integral invertible matrix P such that the $(1,1)$ entry of $P'DP$ is non-zero. Use this matrix P to change our basis $\{e_1, \ldots, e_{i-1}, e_i, \ldots, e_n\}$ to $\{e_1, \ldots, e_{i-1}, e'_i, \ldots, e'_n\}$. In this new basis the matrix for q is obtained by conjugating the q matrix by $\begin{pmatrix} I & O \\ O & P \end{pmatrix}$, which yields $q = \begin{pmatrix} A & BP \\ P'B' & P'CP \end{pmatrix}$. We claim that q restricted to the subspace spanned by $\{e_1, \ldots, e_i\}$ is non-singular. To verify this it suffices to apply the "same" *rational* change of basis we used above but only to the first $i \times i$ submatrix. This yields $\begin{pmatrix} A & O \\ O & (P'DP)_{11} \end{pmatrix}$ where 11 means the $(1,1)$ entry. Since this matrix is non-singular, the original q restricted to the span of $\{e_1, \ldots, e_i\}$ is non-singular. $\square$

Now apply 4.3 to the linking matrix V. The change of bases can be achieved by reordering the handles and by "handle slides" in M_0. Thus we may assume that V_i, the linking matrix of $\{\gamma_1, \ldots, \gamma_i\}$ with respect to M_0, is non-singular for each $1 \leq i \leq n$. Let M_i^* be the result of the surgeries on $\{\gamma_1, \ldots, \gamma_i\}$. We can assume by induction that the cobordism W^* from M_0 to M_i^* is a product on H_1 modulo torsion. Thus $[\gamma_{i+1}]$ is of finite order in $H_1(M_i^*; \mathbf{Z})$ since it is of finite order in $H_1(M_0; \mathbf{Z})$. By the argument of the proof of 4.2, $H_1(M_{i+1}^*; \mathbf{Q}) \cong H_1(M_0; \mathbf{Q})$ since V_{i+1} is non-singular. But then $H_1(M_{i+1}^*; \mathbf{Q}) \cong H_1(M_i^*; \mathbf{Q})$ so the surgery on $\gamma_{i+1} \subseteq M_i^*$ is non-longitudinal with respect to M_i^*. Thus M_1 is N-surgery related to M_0.

This concludes the proof of Theorem 4.2. $\square$

We can now prove 2.9.

Proof of 2.9. Suppose M_0 and M_1 are rationally 2-surgery equivalent. By 2.3 we may assume that there is a sequence $M_0 = X_0 \to X_1 \to \cdots \to X_m = M_1$ where X_{i+1} is obtained from X_i by a single integral non-longitudinal surgery on a rationally null-homologous circle γ_{i+1}. We may assume $\{\gamma_i\}$ are disjoint in M_0. Consider the induced

cobordism W from M_0 to X_i discussed in the proof of 2.3. The proof of 2.3 shows that W is a product on H_1 modulo torsion. Since $[\gamma_{i+1}]$ is trivial in $H_1(X_i;\mathbf{Q})$, it is also trivial in $H_1(M_0;\mathbf{Q})$. Moreover the argument above in the proof of 4.2 shows that the linking matrix of $\{\gamma_1,\dots,\gamma_m\}$ in M_0 is non-singular over $\mathbf{Q}$. $\square$

We want to show that there exist maps $f_i : M_i \to X$ which induces isomorphisms on the first integer homology group and such that $(f_0)_*([M_0]) = (f_1)_*([M_1]) \in H_3(X)$. We may assume as before that f_0 is the inclusion map. Here $X = K\big(H_1(M_0),1\big)$. Of course, since X is aspherical, there exists a map f_1 induced by ϕ_1.

First we note that it would suffice to show that $\langle f_0^*(k),[M_0]\rangle = \langle f_1^*(k),[M_1]\rangle$ for *certain* $h \in H^3(X;\mathbf{Z}_n)$. For if $\alpha = (f_0)_*([M_0]) - (f_1)_*([M_1])$ is not zero in $H_3(X;\mathbf{Z})$ then there is an element of $\mathrm{Hom}(H_3(X);\mathbf{Z}_n)$ which detects it, since $H_3(X)$ is a finitely generated abelian group which has a element of order n only if $H_1(M_0)$ has an element of order n. More precisely, suppose $H_1(M_0) \cong \mathbf{Z}^m \times \mathbf{Z}_{n_1} \times \ldots \times \mathbf{Z}_{n_k}$ where each n_i is a prime power. Then the torsion-free summand of $H_3\big(H_1(M_0);\mathbf{Z}\big)$ is merely $H_3(\mathbf{Z}^m;\mathbf{Z}) \cong H_3(S^1\times\ldots\times S^1;\mathbf{Z})$. If α lies in this summand then it can be detected by an element h of the subgroup $H^3(\mathbf{Z}^m;\mathbf{Z})$. On the other hand if α is of finite order, then it can be detected by some $h \in H^3(X;\mathbf{Z}_n)$ where $n = p^r$ where p^r is the *maximal* order of all elements in $H_1(M_0)$ which have order a power of p. This is true since $\mathbf{Z}_{p^i}$ injects into $\mathbf{Z}_{p^r}$ if $r \geq i$. Therefore these are the only types of elements h we need consider.

Next we need to understand the cohomology rings $H^*(X;\mathbf{Z}_n)$.

Proposition 4.4. *Suppose X is a finitely generated abelian group and n is the exponent of the p-torsion subgroup of X (elements of order p^i). Then the ring $H^3(X;\mathbf{Z}_n)$ is generated by elements of the form $\alpha\cup\beta\cup\gamma$ and $\alpha\cup\tau_*B(\gamma)$ where α, β, $\gamma \in H^1(X;\mathbf{Z}_n)$, B is the Bockstein associated to $0 \longrightarrow \mathbf{Z} \xrightarrow{n} \mathbf{Z} \xrightarrow{\tau} \mathbf{Z}_n \longrightarrow 0$, and $\tau_* : H^2(X;\mathbf{Z}) \longrightarrow H^2(X;\mathbf{Z}_n)$.*

Before proving 4.4, we finish the proof that $C \Rightarrow B$. First consider the case that

$h = \alpha \cup \beta \cup \gamma$. Note that this includes the case $n = 0$ since $H^3(\mathbf{Z}^m; \mathbf{Z})$ is generated by such elements. Then we have (using C b)),

$$\begin{aligned}
\langle f_0^*(h), [M_0] \rangle &= \langle f_0^*(\alpha) \cup f_0^*(\beta) \cup f_0^*(\gamma), [M_0] \rangle \\
&= \left\langle \phi_n^1 \circ f_0^*(\alpha) \cup \phi_n^1 \circ f_0^*(\beta) \cup \phi_n^1 \circ f_0^*(\gamma), [M_1] \right\rangle \\
&= \langle f_1^*(h), [M_1] \rangle
\end{aligned}$$

since $\phi_n^1 \equiv f_1^* \circ (f_0^*)^{-1}$. Now suppose $h = \alpha \cup \tau_* B(\gamma)$. $f_0^*\big(\alpha \cup \tau_* B(\gamma)\big) = f_0^*(\alpha) \cup f_0^* \tau_* B(\gamma) = f_0^*(\alpha) \cup \tau_* B f_0^*(\gamma)$ since f_0 is a continuous map. Hence, using condition c of C we have that:

$$\begin{aligned}
\langle f_0^*(h), [M_0] \rangle &= \left\langle \phi_n^1 \circ f_0^*(\alpha) \cup \tau_* B\left(\phi_n^1 \circ f_0^*(\gamma)\right), [M_1] \right\rangle \\
&= \left\langle f_1^*(\alpha) \cup \tau_* B\big(f_1^*(\gamma)\big), [M_1] \right\rangle \\
&= \langle f_1^*(h), [M_1] \rangle .
\end{aligned}$$

Thus 4.4 will complete $C \Rightarrow B$.

Proof of 4.4. First we need the following:

Lemma 4.5. *Suppose X and Y are spaces whose homology is finitely generated in each dimension. Suppose n is a prime power. Then the cohomology cross product induces an isomorphism:*

$$\Theta_n : \sum_{p+q=3} H^p(X; \mathbf{Z}_n) \oplus_{\mathbf{Z}_n} H^q(Y; \mathbf{Z}_n) \longrightarrow H^3(X \times Y; \mathbf{Z}_n).$$

Proof of 4.5. Θ_n is a monomorphism by [Mu; Theorem 61.6]. Since the homology groups of X and Y are finitely generated the domain and range of Θ_n are finite groups. Thus it will suffice to show that they are *abstractly* isomorphic. First we list some abbreviations: $x_p \equiv H_p(X; \mathbf{Z})$, $y_q \equiv H_q(Y; \mathbf{Z})$, $x_p^t \equiv \mathbf{Z}_n$-torsion subgroup of x_p, $e_p^x = \operatorname{Ext}(x_p; \mathbf{Z}_n)$, $\otimes \equiv \otimes_{\mathbf{Z}}$, $\otimes_n \equiv \otimes_{\mathbf{Z}_n}$. If A, B are finitely generated abelian groups, then the following are easily verified: $\operatorname{Hom}(A; \mathbf{Z}_n) \cong A \otimes \mathbf{Z}_n$, $e_p^x = x_p^t \otimes \mathbf{Z}_n \cong x_p^t$, $A * B \cong A^t * B^t$,

$(A \otimes \mathbf{Z}_n) \otimes_n (B \otimes \mathbf{Z}_n) \cong (A \otimes B) \otimes \mathbf{Z}_n$. Expanding $H^3(X \times Y; \mathbf{Z}_n)$ using the Universal Coefficient Theorem for cohomology and then the Kunneth Theorem for homology and applying the above, we get $\bigoplus_{p+q=3}(x_p \oplus y_q) \oplus (x_1^t * y_1^t) \oplus x_2^t \oplus y_2^t \oplus \mathrm{Ext}(x_1 \oplus y_1; \mathbf{Z}_n)$ all tensored with $\mathbf{Z}_n$. On the other hand the domain of Θ_n may be expanded as $\bigoplus_{p+q=3}\left[\left(x_p \oplus \mathbf{Z}_n \oplus x_{p-1}^t\right) \oplus_n \left(y_q \oplus \mathbf{Z}_n \oplus y_{q-1}^t\right)\right]$. Expanding and comparing terms shows that these expressions are isomorphic, using the fact that $\mathrm{Ext}(x_1 \oplus y_1; \mathbf{Z}_n) \oplus x_1^t \oplus y_1^t \cong (x_1^t \oplus_n (y_1 \oplus \mathbf{Z}_n)) \oplus ((x_1 \oplus \mathbf{Z}_n) \oplus_n y_1^t)$ which is easily seen by expressing x_1, y_1 as direct sums of their "torsion and torsion-free parts."

Now we show that 4.5 implies 4.4. Since X is a finitely-generated abelian group, it is a product $\times_{i=1}^k X_i$ cyclic groups of infinite or prime-power order. Now apply 4.5 inductively. Recall that if $\alpha \in H^p(X; \mathbf{Z}_n)$, $\beta \in H^q(Y; \mathbf{Z}_n)$ then $\alpha \times \beta = \pi_1^*(\alpha) \cup \pi_2^*(\beta)$ where π_i are the projections to the factors. Then 4.5 implies that $H^3(X; \mathbf{Z}_n)$ is generated by elements of the form $\pi^*(\alpha) \cup \pi^*(\beta) \cup \pi^*(\gamma)$, $\pi^*(\alpha) \cup \pi^*(\Delta)$, and $\pi^*(\Gamma)$ where $\alpha \in H^1(X_i; \mathbf{Z}_n)$, $\beta \in H^1(X_j; \mathbf{Z}_n)$, $\gamma \in H^1(X_k; \mathbf{Z}_n)$, $\Delta \in H^2(X_s; \mathbf{Z}_n)$, $\Gamma \in H^3(X_m; \mathbf{Z}_n)$. The only cases where $H^2(X_s; \mathbf{Z}_n)$ is non-zero are $H^2(\mathbf{Z}_{p^s}; \mathbf{Z}_{p^r})$ where $s \leq r$ since n is the exponent. Consider the coefficient sequence $0 \longrightarrow \mathbf{Z}_{p^r} \xrightarrow{i} \mathbf{Z}_{p^{2r}} \xrightarrow{\pi} \mathbf{Z}_{p^r} \longrightarrow 1$. Then the induced map $H^1(\mathbf{Z}_{p^s}; \mathbf{Z}_{p^{2r}} \xrightarrow{\pi_*} H^1(\mathbf{Z}_{p^s}; \mathbf{Z}_{p^r})$ is zero since the composition $\mathbf{Z}_{p^s} \xrightarrow{\phi} \mathbf{Z}_{p^{2r}} \xrightarrow{\pi} \mathbf{Z}_{p^r}$ is zero for any ϕ. Hence the Bockstein $\widetilde{B} : H^1(\mathbf{Z}_{p^s}; \mathbf{Z}_{p^r}) \longrightarrow H^2(\mathbf{Z}_{p^s}; \mathbf{Z}_{p^r})$ is an isomorphism and consequently $\Delta = \widetilde{B}(\gamma)$ for some $\gamma \in H^1(X_s; \mathbf{Z}_n)$ and $\pi^*(\alpha) \cup \pi^*(\Delta)$ is $\pi^*(\alpha) \cup \widetilde{B}\pi^*(\gamma)$. But in this case $\tau_* : H^2(\mathbf{Z}_{p^s}; \mathbf{Z}) \longrightarrow H^2(\mathbf{Z}_{p^s}; \mathbf{Z}_{p^r})$ is an isomorphism and so one sees that $\widetilde{B} = \tau_* B$ as desired.

The only cases where $H^3(X_m; \mathbf{Z}_n)$ is non-zero are $H^3(\mathbf{Z}_{p^m}; \mathbf{Z}_{p^r})$ where $m \leq r$. We shall show $\Gamma = \alpha \cup \Delta$ reducing to the case above. Let $L = L(p^m, 1)$ be the 3-dimensional lens space. The map $L \xrightarrow{i} K(\mathbf{Z}_{p^m}, 1)$ can be constructed by adding cells of dimension 4 and higher to L and thus induces isomorphisms on first and second cohomology and a monomorphism on H^3. By Poincaré Duality for L, $i^*(\Gamma) = i^*(\alpha) \cup i^*(\Delta)$ for some

$\alpha \in H^1(\mathbf{Z}_{p^m};\mathbf{Z}_{p^r})$ and $\Delta \in H^2(\mathbf{Z}_{p^m};\mathbf{Z}_{p^r})$. Hence $\Gamma = \alpha \cup \Delta$ as claimed. This completes the verification 4.5⇒4.4. $\square$

D ⇒ C: For brevity let ϕ_* denote the isomorphism ϕ_1 and ϕ^* denote its induced adjoint ϕ^1_n. Let i denote the inclusion $\mathbf{Z}_n \longrightarrow \mathbf{Q}/\mathbf{Z}$ where $1 \longrightarrow \frac{1}{n}$. Let $T_i(M)$ denote the torsion subgroup of $H_i(M;\mathbf{Z})$. Since a linking pairing λ is non-singular, $\lambda(-,a)$ is an isomorphism $T_1(M) \longrightarrow \mathrm{Hom}(T_1(M);\mathbf{Q}/\mathbf{Z})$. For any $\gamma \in H^1(M;\mathbf{Z}_n)$, $i\,\langle\gamma,-\rangle \in \mathrm{Hom}(T_1(M);\mathbf{Q}/\mathbf{Z})$ and we let $D(\gamma)$ be its inverse under the above isomorphism. Thus $D : H^1(M;\mathbf{Z}_n) \longrightarrow T_1(M)$ and $\lambda(D(\gamma),a) = i\,\langle\gamma,a\rangle$ for all $a \in T_1(M)$.

We claim that if $\gamma \in H^1(M_1;\mathbf{Z}_n)$ then $\phi_* D_0 \phi^*(\gamma) = D_1(\gamma)$ where D_0, D_1 correspond to M_0, M_1 respectively. For if $a \in T_1(M_1)$ then $a = \phi_*(b)$ for some $b \in T_1(M_0)$. So $\lambda_1(\phi_* D_0 \phi^*(\gamma), a) = \lambda_1(\phi_* D_0 \phi^*(\gamma), \phi_*(b)) = \lambda_0(D_0\phi^*(\gamma), b)$, by hypothesis c) 4.1 D. Continuing, $\lambda_0(D_0\phi^*(\gamma), b) = i\,\langle\phi^*(\gamma), b\rangle = i\,\langle\gamma,\phi_*(b)\rangle = i\,\langle\gamma,a\rangle$. Hence $\phi_* D_0\phi^*(\gamma) = D_1(\gamma)$.

Next we claim that the Poincaré dual of $B(\gamma)$ is $D_1(\gamma)$ where $B : H^1(M_1;\mathbf{Z}_n) \longrightarrow H^2(M_1;\mathbf{Z})$ is the Bockstein associated to $0 \longrightarrow \mathbf{Z} \xrightarrow{\cdot n} \mathbf{Z} \longrightarrow \mathbf{Z}_n \longrightarrow 0$. We need to show $\lambda_1(B(\gamma)\cap[M_1], a) = i\,\langle\gamma,a\rangle$ for each $a \in T_1(M_1)$. Suppose g is 1-cochain with coefficients in $\mathbf{Z}_n$ representing γ and $\tilde{g}$ is an integral 1-cochain reducing to g. Then $B(\gamma)$ is represented by $\frac{1}{n}\delta\tilde{g}$, a 2-cochain with integral values. Note that $B(\gamma)$ is n-torsion so its Poincaré dual lies in $T_1(M_1)$. If Σ is a chain representing the orientation class $[M_1]$ then the Poincaré dual is represented by $\frac{1}{n}\delta\tilde{g}\cap\Sigma = \frac{1}{n}\partial(\tilde{g}\cap\Sigma)$. Thus $\lambda_1(B(\gamma)\cap[M],a)$ is given by $\frac{1}{n}\cdot\#\big((\tilde{g}\cap\Sigma)\cdot a'\big)$ where a' is a chain representing a and $\#$ is the number of signed intersection points modulo n. But this is also a calculation of $i\,\langle\gamma,a\rangle$, finishing the verification of our second claim.

Now we can finish the proof that $D \Rightarrow C$. If α, $\gamma \in H^1(M_1; \mathbf{Z}_n)$ then

$$\begin{aligned} i\,\langle \alpha \cup \pi^* B\gamma, [M_1]\rangle &= i\,\langle \alpha, \pi^* B\gamma \cap [M_1]\rangle \\ &= i\,\big\langle \alpha, \pi_*\big(B(\gamma) \cap [M_1]\big)\big\rangle \\ &= i\,\langle \alpha, \pi_* D_1(\gamma)\rangle \\ &= i\,\langle \alpha, D_1(\gamma)\rangle \\ &= \lambda_1\,(D_1(\alpha), D_1(\gamma)) \end{aligned}$$

and similarly

$$\begin{aligned} i\,\langle \phi^*\alpha \cup \pi^* B\phi^*\gamma, [M_0]\rangle &= \lambda_0\big(D_0(\phi^*\alpha), D_0(\phi^*(\gamma))\big) \\ &= \lambda_1\big(\phi_* D_0 \phi^*\alpha, \phi_* D_0 \phi^*\gamma\big) \end{aligned}$$

by hypothesis D. By our first claim this equals $\lambda_1\big(D_1(\alpha), D_1(\gamma)\big)$ as above. Since i is injective, condition b) of C is established. $\square$

§**5. Rational Homology Surgery Equivalence.** In this chapter we address the question of when two 3-manifolds are related by a sequence of Dehn surgeries on rationally null-homologous curves which preserve $H_1(\_;\mathbf{Q})$. In the language of §2, this is case iv) where $N = G_2^{\mathbf{Q}} = \{x \in G \mid \exists n, x^n \in G_2\}$, or *rational* 2*-surgery equivalence*. Note that $G/G_2^{\mathbf{Q}} = H_1(G)/T_1(G)$. By 2.2 this is an equivalence relation and by 2.3 it is sufficient to consider non-zero *integral* framings. We find that this is completely controlled by the isomorphism class of the integral cup product form. Beware that, because we restrict to rationally null-homologous curves, a surgery which preserves β_1 will necessarily preserve H_1/T_1 and consequently $H^1(\_;\mathbf{Z})$. Therefore it is not possible to have, for example, $H_1(M_0) \cong \mathbf{Z}$, $H_1(M_1) \cong \mathbf{Z}$ with the natural map between them being "times 2." This would be possible to achieve by allowing certain surgeries on curves in M_0 which are *essential* in $H_1(M_0;\mathbf{Z})$ but not primitive. Hence rational 2-equivalence is NOT the relation generated by Dehn surgeries which preserve $H_1(\_;\mathbf{Q})$ but rather those which preserve $H_1(\_;\mathbf{Z})/T_1$.

Theorem 5.1. *Suppose M_0 and M_1 are closed, oriented connected 3-manifolds. The following 3 conditions are equivalent.*

QA) *M_0 and M_1 are rationally 2-surgery equivalent; that is, each may be obtained from the other by a sequence of non-longitudinal Dehn surgeries on circles which are zero in $H_1(\_;\mathbf{Q})$ (equivalently* **integral** *non-longitudinal surgeries.*

QB) *There exists an isomorphism $\phi_1 : H_1(M_1)/T_1(M_1) \longrightarrow H_1(M_0)/T_1(M_0)$ such that $(f_0)_*([M_0]) = (f_1)_*([M_1])$ in $H_3\big(H_1(M_0)/T_1(M_0);\mathbf{Z}\big) \cong H_3\big((S^1)^{\beta_1(M_0)};\mathbf{Z}\big)$ where f_0 is induced by "inclusion" and f_1 is induced by ϕ_1. That is, M_0 and M_1 are bordant over $(S^1)^{\beta_1(M_0)}$.*

QC) *There exists an isomorphism ϕ_1 as in the first line of $\mathbf{Q}B$ such that $\langle \alpha\cup\beta\cup\gamma, [M_0]\rangle = \big\langle \phi^1(\alpha)\cup\phi^1(\beta)\cup\phi^1(\gamma), [M_1]\big\rangle$ for all α, β, $\gamma \in H^1(M_0;\mathbf{Z})$ and ϕ^1 is the adjoint (Hom-Dual) of ϕ_1. That is, the integral cup product forms of M_0 and M_1 are isomorphic.*

Proof of 5.1. **QA $\Rightarrow$ QB**: By 2.3 with $N = \big(\pi_1(M_0)\big)_2^{\mathbf{Q}}$, we may reduce to the case of a single integral non-longitudinal surgery on $\gamma \subset M_0$ such that $[\gamma] \in N$. If W is the cobordism corresponding to this surgery, then $\beta_1(W) = \beta_1(M_0)$ and the inclusion map induces an isomorphism modulo the N-subgroups (see last paragraph of proof of 2.3), which, in the case at hand, means it induces an epimorphism on H_1 modulo torsion. But by the symmetry of 2.1 and 2.2 the same may be said of M_1. Hence we may let $\phi_1 = (j_0)_*^{-1}\circ(j_1)_*$ as in the proof of 4.1. Then, as in 4.1, there are continuous maps f_0, F and f_1 from M_0, W, M_1 respectively, to $K(\mathbf{Z}^{\beta_1(M_0)}, 1)$ inducing the obvious maps on π_1 and the result follows. $\square$

QA $\Rightarrow$ QC: Note that the inclusion maps j_0, j_1 as defined above induce isomorphisms on $H^1(\_;\mathbf{Z})$ since $\operatorname{Hom}(H_1;\mathbf{Z}) \cong \operatorname{Hom}(H_1/T_1;\mathbf{Z})$. The proof is now the same as the first part of the proof of $A \Rightarrow D$ in §4.

QC $\Rightarrow$ QB: Note that all the maps j_0^*, j_1^*, ϕ^1, f_0^*, f_1^* are isomorphisms on $H^1(\_;\mathbf{Z})$. Since the cohomology ring of $K(\mathbf{Z}^m,1)$ is well-known to be generated by triple cup products, the easy part of the proof of $C \Rightarrow B$ in §4 applies word for word. $\square$

QB $\Rightarrow$ QA: Apply 4.2 with $N = \big(\pi_1(M_0)\big)_2^{\mathbf{Q}}$. $\square$

Let us denote by $\mathcal{S}_m^{\mathbf{Q}}$ the set of rational homology surgery equivalence classes of closed oriented 3-manifolds with $\beta_1 = m$. By 5.1, if $m < 3$ then $\mathcal{S}_m^{\mathbf{Q}}$ contains a single element.

Corollary 5.2. *If $m < 3$ any 2 closed, oriented 3-manifolds with identical first Betti number m are rational homology surgery equivalent.*

Corollary 5.3. *There is a bijection $\mathcal{S}_m^{\mathbf{Q}} \to \Lambda^3(\mathbf{Z}^m)/\operatorname{GL}_m(\mathbf{Z})$ given by the integral triple cup product form. Hence if M_0, M_1 have torsion-free homology groups then they are rational homology surgery equivalent if and only if they are integral homology surgery equivalent.*

Proof of 5.3. See the proof of 3.5 and use Sullivan's work [Su].

Example 5.4: It is possible for M_0 and M_1 to be rational homology surgery equivalent, have isomorphic first homology and linking forms, yet not be integral homology surgery equivalent (see Example 3.15).

Corollary 5.5. *For any closed, oriented 3-manifold M, M is rational homology surgery equivalent to $-M$.*

§**6. Surgery Equivalence Preserving Lower Central Series Quotients.** We have seen that the relation generated by $\pm 1/n$ Dehn surgery on circles which lie in $(\pi_1(M))_k$ is an equivalence relation which we called k-surgery equivalence. The equivalence relation generated by non-longitudinal surgeries on circles lying in the k^{th} term of the rational lower central series, we call rational k-surgery equivalence. Just as 2-equivalence was controlled by G/G_2 and the cup products (and linking form), we shall see that k-equivalence is controlled by G/G_k and higher Massey products (and the linking form). We only attempt a complete algebraic characterization of k-surgery equivalence to "the zero element," i.e. $\#_{i=1}^m S^1 \times S^2$. Here we see that k-equivalence is controlled by Massey products of length less than $2k-1$, or equivalently by the isomorphism class of G/G_{2k-1}. A similar characterization for the general case is made difficult by the ill-definedness of Massey products and our ignorance of H_3 of torsion-free nilpotent groups. It may well be, however, that there is sufficient information in the literature to complete the general characterization.

Theorem 6.1. *Suppose M_0 and M_1 are closed, oriented, connected 3-manifolds. The following are equivalent.*

A) *M_0 and M_1 are k-surgery equivalent.*
B) *There exists an isomorphism $\phi : \pi_1(M_1)/\big(\pi_1(M_1)\big)_k \longrightarrow \pi_1(M_0)/\big(\pi_1(M_0)\big)_k$ such that $\phi_*([M_1]^k) = [M_0]^k$ where $[M_i]^k$ means the image, in $H_3\big(\pi_1(M_i)/\big(\pi_1(M_i)\big)_k;\mathbf{Z}\big)$, of the fundamental class of M_i under some map $f_i : M_i \to K\big(\pi_1(M_i)/\big(\pi_1(M_i)\big)_k, 1\big)$ inducing the obvious quotient on π_1.*

Corollary 6.2. *The set of k-surgery equivalence classes of closed, oriented 3-manifolds M with $\pi_1(M)/\big(\pi_1(M)\big)_k \cong G$ is in bijection with the subset of $H_3(G)/\operatorname{Aut}(G)$ consisting of those elements which are "realizable", that is which can arise as $[M]^k$ for* **some** *closed 3-manifold. The correspondence is given by the fundmental class (see* [Tu1] *for an analysis of this realizable set.*

Proof of 6.1. $A \Rightarrow B$ is implied by Propositions 2.3, 2.1 and 4.1.

$B \Rightarrow A$ is implied by Theorem 4.2. $\square$

Proof of 6.2. Merely note that $[M_i]^k$ is only well-defined up to the action of $\operatorname{Aut}(G_i)$ on $H_3(G_i)$.

Theorem 6.3. *Suppose M_0 and M_1 are closed, oriented, connected 3-manifolds. The following are equivalent.*

QA) *M_0 and M_1 are rationally k-surgery equivalent.*

QB) *Same condition as 6.1B with rational lower central series replacing the integral one.*

Corollary 6.4. *The set of rational k-surgery equivalence classes of closed, oriented 3-manifolds with $\pi_1(M)/\big(\pi_1(M)\big)_k^{\mathbf{Q}} \cong G$ is in bijection with the subset of $H_3(G)/\operatorname{Aut}(G)$ corresponding to realizable classes (see* [Tu1]*).*

Proof of 6.4. The argument for $\mathbf{Q}A \Rightarrow \mathbf{Q}B$ in the proof of 5.1 works here (using that finitely generated nilpotent groups are Hopfian). For $\mathbf{Q}B \Rightarrow \mathbf{Q}A$ apply 4.2 with $N = \big(\pi_1(M_0)\big)_k^{\mathbf{Q}}$. $\square$

Corollary 6.5. *Any two 3-manifolds with $\beta_1 = 0$ (or any two with $\beta_1 = 1$) are rationally k-surgery equivalent for each k.*

Proof of 6.5. Suppose $\beta_1(M_0) = 1$. Then the epimorphism $\pi_1(M_0) \twoheadrightarrow \mathbf{Z}$ induces isomorphisms modulo any term of the rational lower central series [St], so $\pi_1(M_0)/\big(\pi_1(M_0)\big)_k^{\mathbf{Q}}$ is $\mathbf{Z}$. Since $H_3(\mathbf{Z}) = 0$, the result follows from 6.4. $\square$

Example 6.6: Let $M_0 = S^1 \times S^2 \# S^1 \times S^2$ and let M_1 be the manifold obtained by 0-surgery on each component of a Whitehead link in S^3, as shown by the solid lines in Figure 6.7a. Performing a $+1$ surgery on the dashed circle γ in Figure 6.7b transforms M_0

Figure 6.7

to M_1, and γ is clearly null-homologous in M_0 so M_0 and M_1 are 2-surgery equivalent (as they must be since $H_3(\mathbf{Z}\times\mathbf{Z})=0$). However $\gamma\notin\big(\pi_1(M_0)\big)_3$ and so it is not clear whether or not M_0 and M_1 are 3-surgery equivalent. In this case M_1 is known to be the "Heisenberg manifold" (Euler class ± 1 circle bundle over the torus) whose fundamental group is F/F_3 where F is the free group on $\{x,y\}$. Hence $\pi_1(M_0)/\big(\pi_1(M_0)\big)_3 \cong \pi_1(M_1)/\big(\pi_1(M_1)\big)_3 \cong F/F_3$. But 6.1B is not satisfied since $[M_0]$ represents the trivial class in $H_3(F/F_3)$ since it factors through $H_3(F)$, whereas M_1 is a $K(F/F_3,1)$ and so $[M_1]$ represents a generator of $H_3(F/F_3)\cong\mathbf{Z}$. Thus the manifolds are not 3-surgery equivalent (nor rationally 3-surgery equivalent).

We shall now show that k-surgery equivalence is related to higher order Massey products and that this is the correct generalization of the triple cup product form. However, Massey products may not be uniquely defined and this makes statements of results difficult. For this reason we shall restrict our focus to situations where the Massey products are uniquely defined. In general if M_0 and M_1 are k-surgery equivalent then their lower central series quotients G/G_j are isomorphic for $1\le j\le k$ and this is known to entail a "correspondence" between order $k-1$ Massey products with any abelian coefficients [Dw; Corollary 2.7]. We shall state this only for a restricted case.

Proposition 6.8. *Suppose M_0 and M_1 are k-surgery equivalent, and that all j^{th} order Massey products vanish for M_0 for $2\le j\le(k-2)$. Then there is an isomorphism $\phi: H_1(M_1)\to H_1(M_0)$ such that for all abelian groups A and $\alpha_i\in H^1(M_0;A)$, $1\le i\le k$, $\langle\alpha_1\cup\langle\alpha_2,\ldots,\alpha_k\rangle,[M_0]\rangle=\langle\phi^*\alpha_1\cup\langle\phi^*\alpha_2,\ldots,\phi^*\alpha_k\rangle,[M_1]\rangle$ where $\langle\alpha_2,\ldots,\alpha_k\rangle$*

is the Massey product in $H^2(M_0;A)$. In fact, $\pi_1(M_0)/\big(\pi_1(M_0)\big)_k \cong F/F_k$ (F a free group) if and only if $H_1(M_0)$ is torsion free and all Massey products of length less than k vanish for M_0. More precisely, the latter conditions imply that any given isomorphism $\pi_1(M_0)/\big(\pi_1(M_0)\big)_{k-1} \cong F/F_{k-1}$ can be extended.

FIGURE 6.10

Example 6.9: Let $M_0 = \#_{i=1}^4 S^1 \times S^2$ and let M_1 be the manifold shown in Figure 6.10a. as 0-surgery on the 4-component link L. Then M_0 is 2-surgery equivalent to M_1 as shown by the 3 circles labelled ± 1 in 6.10b.. Note these circles lie in the $G_2 - G_3$ where $G = \pi_1(M_0)$. It is known that the link in 6.10a. has $\overline{\mu}(1234) = \pm 1$, so for M_0 all Massey products vanish, but for M_1, $\langle x_1 \cup \langle x_2, x_3, x_4\rangle, [M_1]\rangle = \pm 1$ where x_i are the Hom-duals of the meridians [Ko; Theorem 3]. Hence M_0 and M_1 are not 4 surgery equivalent, by 6.8. In fact they are not even 3-surgery equivalent but 6.8 is too weak to show this.

Proof of 6.8. Apply [Dw; Corollary 2.7] to the maps $j_0 : M_0 \to W$ and $j_1 : M_1 \to W$ where W is the cobordism over $\pi_1(M_0)/\big(\pi_1(M_0)\big)_k$ guaranteed by 6.1. Then use naturality of Massey products and $(j_0)_*([M_0]) = (j_1)_*([M_1])$ to get the first claimed result. Details are left for the reader.

We consider the last claim of 6.8. Suppose $\pi_1(M_0)/\big(\pi_1(M_0)\big)_k \cong F/F_k$, $k \geq 2$. Then there is a map $f : M_0 \to K(F/F_k, 1)$ which induces an isomorphism on first cohomology with any abelian coefficients. It is known that F/F_k has vanishing Massey products of length less than k and that $H^2(F/F_k)$ is generated by Massey products of length k [O2;

Lemma 16]. By naturality, $f^*\big(\langle x_1,\dots,x_m\rangle\big) \subseteq \langle f^*x_1,\dots,f^*x_m\rangle$. Hence, if $m<k$ then $0\in\langle f^*x_1,\dots,f^*x_m\rangle$. But the first non-vanishing level of Massey products are uniquely defined, so by induction, $0=\langle f^*x_1,\dots,f^*x_m\rangle$ for $m<k$. This implies that all Massey products of length less than k vanish for M_0.

Now suppose $H_1(M_0)$ is torsion-free and all Massey products of length less than k vanish for M_0. By induction we can assume $\pi_1(M_0)/\big(\pi_1(M_0)\big)_{k-1}\cong F/F_{k-1}$ so there is a map $g: F\to G$ ($\pi_1(M_0)=G$) inducing this isomorphism. It would then suffice to prove

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & H_2(F/F_{k-1}) \arrow[r] \arrow[d, "g_*", "\cong"'] & F_{k-1}/F_k \arrow[r] \arrow[d, "g_*"] & 0 \\
 & H_2(G) \arrow[r, "\pi_*"] & H_2(G/G_{k-1}) \arrow[r] & G_{k-1}/G_k \arrow[r] & 0
\end{tikzcd}$$

that $F_{k-1}/F_{k-2}\cong G_{k-1}/G_k$ and the diagram above shows that this is equivalent to showing π_* is the zero map. Since $H_2(M_0)$ maps surjectively to $H_2(G)$ it suffices to show $H_2(M_0)\xrightarrow{\pi_*}H_2(G/G_{k-1})$ is the zero map. But $H^2(G/G_{k-1})\cong H^2(F/F_{k-1})$ is generated by $\langle x_1,\dots,x_{k-1}\rangle$ so $\pi^*\langle x_1,\dots,x_{k-1}\rangle=\langle \pi^*x_1,\dots,\pi^*x_{k-1}\rangle=0$. Thus π^* and π_* are zero. Note that we have actually proved that any isomorphism $g_*: G/G_{k-1}\longrightarrow F/F_{k-1}$ can be extended to $g_*: G/G_k\longrightarrow F/F_k$. $\square$

Example 6.9 indicates that 6.8 is too weak. Indeed, 6.1B should be seen as two conditions, and 6.8 says that the first of these conditions controls Massey products of lengths up to $k-1$. We shall see that the second conditions controls lengths up to $2k-2$. This is analagous to the Cochran-Orr conjecture that a link in S^3 is "null k-cobordant" if and only if its Milnor $\overline{\mu}$-invariants of length j, $1\le j\le 2k$ are zero. This has been positively resolved by X.S. Lin and Orr-Igusa [L] [IO]. Based on techniques of the latter, we shall now discuss an algebraic characterization of k-surgery equivalence to $\#S^1\times S^2$.

Theorem 6.10. *For any integer $k\ge 2$, M is k-surgery equivalent to $\#_{i=1}^m S^1\times S^2$ if and only if $H_1(M)\cong\mathbf{Z}^m$ and all Massey products of order* **less** *than $2k-1$ vanish for M. In*

particular if M is zero surgery on an m-component link L in a homology 3-sphere then M is k-surgery equivalent to $\#_{i=1}^m S^1 \times S^2$ if and only if Milnor's $\overline{\mu}$-invariants of length **less** *than $2k$ vanish for L.*

Proof of 6.10. Let $G = \pi_1(M)$. Suppose $H_1(M) \cong \mathbf{Z}^m$ and all Massey products of length less than $2k-1$ vanish for M. By the proof of the last part of 6.8, any isomorphism $g_* : G/G_k \longrightarrow F/F_k$ extends to an isomorphism $h_* : G/G_{2k-1} \longrightarrow F/F_{2k-1}$. It follows that "the" natural map $M \xrightarrow{\pi} K(G/G_k, 1) \xrightarrow{g} K(F/F_k, 1)$ factors through $K(F/F_{2k-1}, 1)$ and thus that $[M]^k = 0$ in $H_3(F/F_k; \mathbf{Z})$ since Igusa and Orr have shown that the map $H_3(F/F_{2k-1}) \longrightarrow H_3(F/F_k)$ is zero [IO]. Hence, by 6.1, M is k-surgery equivalent to $\#_{i=1}^m S^1 \times S^2$.

Now suppose M is k-surgery equivalent to $\#_{i=1}^m S^1 \times S^2$. Then $[M]^k = 0$ in $H_3(F/F_k)$. First we show that this implies that $G/G_{k+1} \cong F/F_{k+1}$. This follows from this more general result.

Lemma 6.11. *Suppose $\pi_1(M_0) = G_0$, $\pi_1(M_1) = G_1$, $G_0/(G_0)_k \cong F/F_k$, $G_1/(G_1)_{k+1} \cong F/F_{k+1}$. If M_0 is k-surgery equivalent to M_1 then $G_0/(G_0)_{k+1} \cong F/F_{k+1}$ by an isomorphism extending f.*

Proof of 6.11. Consider a cobordism W from M_0 to M_1 which contains only 2-handles and is an F/F_k-cobordism (see 4.2). Let F, F_0, F_1 denote the maps to $K(F/F_k, 1)$ from W, M_0, M_1 respectively. For any cohomology classes $\{x_1, \ldots, x_k\} \subset H^1(M)$, choose $y_i \in H^1(F/F_k)$ so $(F_0)^*(y_i) = x_i$. Then $\langle x_1, \ldots, x_k \rangle = \langle F_0^* y_1, \ldots, F_0^* y_k \rangle = j_0^* \langle F^* y_1, \ldots, F^* y_k \rangle$, where the Massey products are uniquely defined since products of lesser length vanish since all spaces have $G/G_k \cong F/F_k$ (see 6.9). By 6.8, it will suffice to show $\langle F^* y_1, \ldots, F^* y_k \rangle = 0$. Certainly $j_1^* \langle F^* y_1, \ldots, F^* y_k \rangle = 0$ since all Massey products of length k vanish for M_1. To finish we will show that $j_1^* : H^2(W) \longrightarrow H^2(M_1)$ is injective on the image of $H^2(F/F_k)$. Recall that we can assume that W is built from

$M_1 \times [0,1]$ by adding 2-handles whose attaching circles lie in F_k. Thus $H_2(W)$ splits as $H_2(M_1) \oplus H_2(W, M_1)$ where the latter is a free abelian group generated by the cores of the 2-handles "capped-off" by surfaces in M_1. But these latter classes clearly become spherical in $K(F/F_k, 1)$ and hence vanish in $H_2(F/F_k, 1)$. Considering the dual splitting of $H^2(W)$, this implies that the image of $H^2(F/F_k)$ lies in the summand $H^2(M_1) \hookrightarrow H^2(W)$. It follows that j_1^* is injective on this image. $\square$

Suppose $\pi_1(M) \cong G$ and $G/G_k \cong F/F_k$ for some free group F. Suppose also that f is a specific such isomorphism. Then we can define $\theta_k(M, f) \in H_3(F/F_k)$ to be the image of the fundamental class under the map $M \longrightarrow K(F/F_k, 1)$ induced by f.

Lemma 6.12. *$\theta_k(M, f) \in \mathrm{Image}\big(H_3(F/F_{k+1}) \xrightarrow{\pi_*} H_3(F/F_k)\big)$ if and only if there is some isomorphism $\tilde{f} : G/G_{k+1} \longrightarrow F/F_{k+1}$ extending f such that $\pi_*\big(\theta_{k+1}(M, \tilde{f})\big) = \theta_k(M, f)$.*

Proof of 6.12. Suppose $\theta_k(M, f) = \pi_*(x)$ $x \in H_3(F/F_{k+1})$. By [O1; Theorem 4], there exists a 3-manifold M_1 with $\pi_1(M_1) \cong G_1$ and $G_1/(G_1)_k \overset{g}{\cong} F/F_{k+1}$ such that $\theta_{k+1}(M_1, g) = x$. Since $\pi_*(x) = \theta_k(M_1, \pi \circ g) = \theta_k(M, f)$, it follows from 6.1 that (M, f) and $(M_1, \pi \circ g)$ are cobordant over F/F_k and are k-surgery equivalent. From 6.11 we can conclude that there is an isomorphism $\tilde{f}$ as desired. Since $\tilde{f}$ extends f, $\pi_*\big(\theta_{k+1}(M, \tilde{f})\big) = \theta_k(M, f)$, by definition. The other implication of 6.12 is immediate. $\square$

To finish the proof of 6.10 we need the following theorem from [IO].

Theorem 6.13 (Igusa-Orr [IO]). *Suppose F is the free group of rank m. Let $N_i = \mathrm{rank}\, H_2(F/F_i)$. Then $H_3(F/F_k; \mathbf{Z})$ is $\bigoplus_{i=k}^{2k-2} \mathbf{Z}^{mN_i - N_{i+1}}$. If we define the* **weight** *of the summand $\mathbf{Z}^{mN_i - N_{i+1}}$ to be $i+1$ then the natural maps $H_3(F/F_{k+1}) \longrightarrow H_3(F/F_k)$ have the property that they preserve weight whenever possible and map each weight summand injectively and by the zero map otherwise.*

Corollary 6.14. *The map $H_3(F/F_{2k-1}) \longrightarrow H_3(F/F_k)$ is zero. Any element in the kernel of $H_3(F/F_{k+j}) \longrightarrow H_3(F/F_k)$, $j \leq k-1$, lies in the image of $H_3(F/F_{2k-1}) \longrightarrow H_3(F/F_{k+j})$.*

Proof of 6.14. An element of the kernel of the above map has weight greater than $2k-1$ and at most $2k+2j-1$ i.e. at most $4k-3$, which is precisely the range of weights possible for an element of $H_3(F/F_{2k-1})$. $\square$

Finally, if $\theta_k(M,f) = 0$ for some f then assume by induction (use 6.12 for the start of induction) that f extends to $\tilde{f} : G/G_{k+j} \longrightarrow F/F_k$ so that $\theta_{k+j}(M,\tilde{f})$ is defined and is a lift of $\theta_k(M,f)$. But then as long as $j \leq k-1$, 6.14 guarantees that $\theta_{k+j}(M,\tilde{f}) \in \text{Image}\big(H_3(F/F_{k+j+1}) \longrightarrow H_3(F/F_{k+j})\big)$ and 6.12 then implies θ_{k+j+1} is defined. Hence we can lift to θ_{2k-1} and $G/G_{2k-1} \cong F/F_{2k-1}$ implying that all Massey products of length less than $2k-1$ are zero for M. This concludes the proof of the first sentence of 6.10. The second statement follows immediately from [Ko; Theorem 2] relating Massey products of 0-surgery on a link to $\overline{\mu}$-invariants of a link. $\square$

It follows from 6.10 that the two manifolds in Example 6.9 are not 3-surgery equivalent.

Example 6.15: The example M_1 in Figure 6.7a (the Heisenberg manifold) is 2-surgery equivalent to $S^1 \times S^2 \# S^1 \times S^2$ since the first non-zero Milnor invariant of the Whitehead link, $\overline{\mu}(1122)$, is of length 4, but for the same reason the Whitehead link is **NOT** null 2-cobordant. In general, if a link is null k-cobordant then its 0-surgery is k-surgery equivalent to $\# S^1 \times S^2$ but the converse is false, requiring in addition that the $\overline{\mu}$-invariants of length $2k$ vanish.

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